Notes on Fourier Transforms

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The Fourier transformation is one of the basic mathematical tools required in dealing with problems of signal correlation. These notes summarize some of the basic aspects of the Fourier transformation, although they do not pretend to be complete in any way. There are many books which deal with the subject at length. The following should be adequate for most of the radio astronomical applications:

1. Fourier Transforms and Convolutions for the Experimentalist, R. C. Jennison, Pergamon, 1961.

2. Transformation de Fourier et Théorie des Distributions, J. Arsac, Dunod, 1961.

3. Fourier Transforms, I. N. Sneddon, McGraw-Hill, 1951.

Extensive tables of Fourier transforms can be found in Volume I of the Bateman Manuscript Project, Tables of Integral Transforms, McGraw-Hill, 1954.

I. Complex Variables

We start with a brief review of the rudiments of complex algebra, since much of the discussion to follow will involve complex quantities.

Consider the complex quantity

$$z = x + jy \tag{1}$$

where $j = \sqrt{-1}$. x and y are respectively the real and imaginary parts of z. It is often convenient to express z in polar form, using its modulus or magnitude r and a phase angle ϑ:

$$z = re^{j\vartheta} \tag{2}$$

where

$$\left.\begin{aligned} r &= \sqrt{x^2 + y^2} \\ \vartheta &= \tan^{-1}\frac{y}{x} \end{aligned}\right\} \tag{3}$$

According to Euler's formula,

$$e^{j\vartheta} = \cos\vartheta + j\sin\vartheta$$

so

$$\left.\begin{aligned} x &= r\cos\vartheta \\ y &= r\sin\vartheta \end{aligned}\right\} \qquad (4)$$

For reference, we note de Moivre's formula:

$$e^{jn\vartheta} = (\cos\vartheta + j\sin\vartheta)^n = \cos n\vartheta + j\sin n\vartheta$$

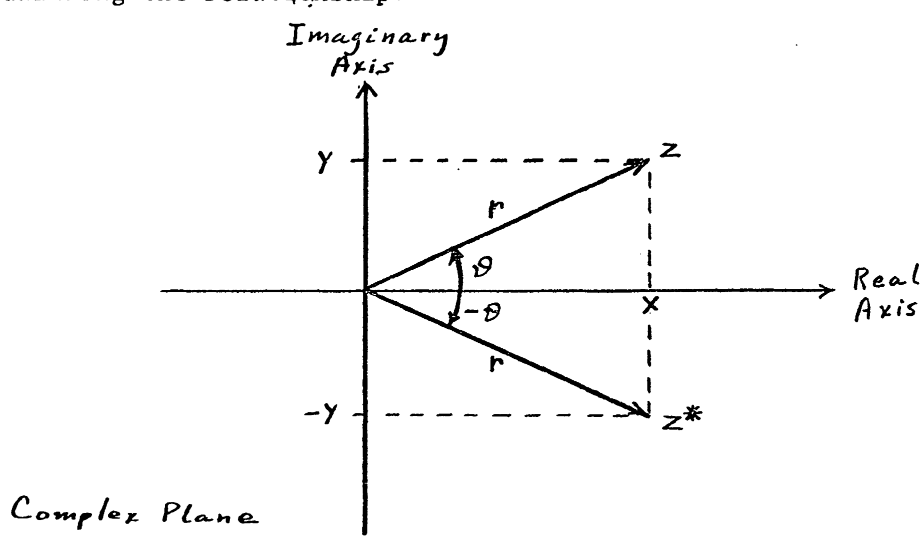
The complex conjugate of z is defined as

$$z^* = x - jy = r\,e^{-j\vartheta} \qquad (5)$$

In general, one obtains the complex conjugate of a complex quantity or expression by inverting the sign of j wherever it occurs. Clearly

$$r^2 = zz^*$$

Note that z can be regarded as a two-dimensional vector of length r with an orientation specified by ϑ. The Argand diagram is a convenient aid in visualizing the relationship:



II. The Fourier Series Representation of a Periodic Function

We shall treat the Fourier series representation in some detail since the basic concepts needed in understanding the Fourier transformation can be developed easily in this way.

Consider a function that repeats itself exactly at intervals of 2π in its argument:

$$f(\vartheta + 2\pi k) = f(\vartheta).$$

If this relation holds for all integral values of k, positive or negative, the function is said to be periodic. A periodic function which repeats at an interval different from 2π can always be forced into the above form by a suitable change of variable. In the following, we shall always assume that the function has 2π as its period.

A function can be represented by a Fourier series if it meets two requirements:

1. It has no more than a finite number of discontinuities in any finite interval of its argument.

2. It is absolutely integrable over a single period; i.e., the integral

   $$\int_c^{c+2\pi} |f(\vartheta)|\, d\vartheta$$

   exists.

Most, if not all, of the functions used to describe physical laws and quantities meet these conditions. Thus they are not very restrictive for practical purposes.

The Fourier series representation of a function $f(\vartheta)$ is usually expressed as

$$f(\vartheta) = a_0/2 + \sum_{n=1}^{\infty} \left\{ a_n \cos n\vartheta + b_n \sin \vartheta \right\} \tag{6}$$

where

$$\left.\begin{aligned} a_0 &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\vartheta)\, d\vartheta \\ a_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\vartheta)\cos n\vartheta\, d\vartheta \\ b_n &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(\vartheta)\cdot \sin n\vartheta\, d\vartheta \end{aligned}\right\} \quad (7)$$

It should be noted that throughout these notes we assume that angles, or quantities treated as angles, are in radians.

The series always converges to the value of the function except at a discontinuity, where it converges to the midpoint of the discontinuity.

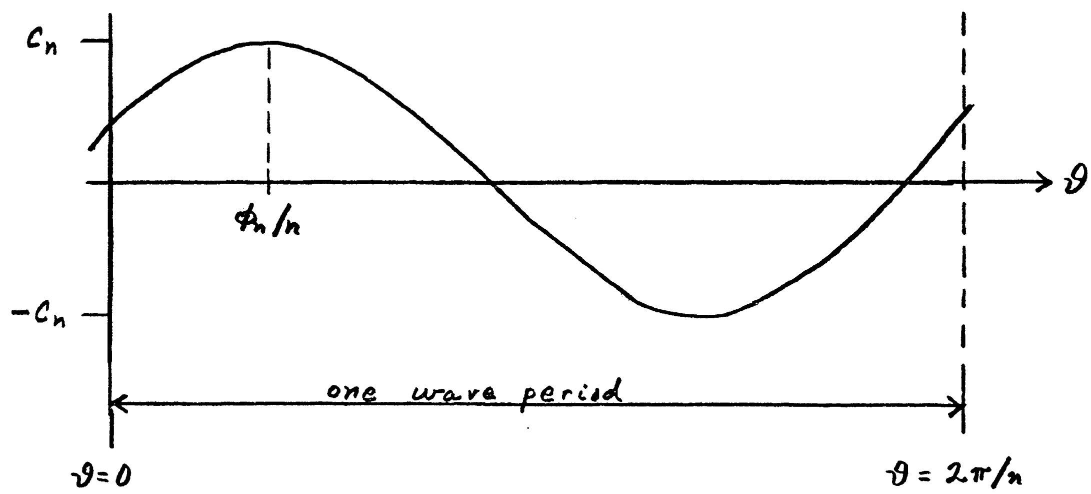
There are alternative ways of writing the Fourier series. The following probably gives the most direct insight into the nature of the representation:

$$f(\vartheta) = a_0/2 + \sum_{n=1}^{\infty} c_n \cos(n\vartheta - \phi_n) \quad (8)$$

where

$$\left.\begin{aligned} c_n &= \sqrt{a_n^2 + b_n^2} > 0 \\ \phi_n &= \tan^{-1} b_n/a_n \end{aligned}\right\} \quad (9)$$

We see that the function is resolved into a "d.c. component" expressed by $a_0/2$, and an infinite number of "a.c. components" at discrete frequencies $n = 1, 2, 3, \ldots$. Here n is the number of "wave periods" in one function period of length 2π. Each "a.c. component" is described by an amplitude c_n, a frequency n, and a phase angle ϕ_n. The diagram illustrates the relationships:



The function $f(\vartheta)$ is the sum of all these waves and the "d.c. component". Each constituent wave is a Fourier component of $f(\vartheta)$.

The Fourier series can be written in complex form. This is the most compact way of expressing it, and it is the form that we shall generalize for non-periodic functions to obtain the Fourier integral representation. From de Moivre's formula, one can readily derive the relations

$$\cos n\vartheta = \tfrac{1}{2}\left(e^{jn\vartheta} + e^{-jn\vartheta}\right)$$

$$\sin n\vartheta = \tfrac{1}{2j}\left(e^{jn\vartheta} - e^{-jn\vartheta}\right)$$

Substituting these in (6), we obtain

$$f(\vartheta) = a_0/2 + \sum_{n=1}^{\infty}\left\{g_n e^{jn\vartheta} + h_n e^{-jn\vartheta}\right\} \tag{10}$$

where

$$\left.\begin{aligned} g_n &= \tfrac{1}{2}(a_n - j b_n) \\ h_n &= \tfrac{1}{2}(a_n + j b_n) = g_n^* \end{aligned}\right\} \tag{11}$$

If we let $g_0 = a_0/2$ and $g_{-n} = h_n$, we can rewrite (10) in the condensed form

$$f(\vartheta) = \sum_{-\infty}^{\infty} g_n e^{jn\vartheta} \tag{12}$$

It can be shown that now

$$q_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\vartheta)\, e^{-jn\vartheta}\, d\vartheta \tag{13}$$

Equation (12) is the complex form of the Fourier series.

We have been calling n the frequency of the nth Fourier component. Thus it seems strange that (12) requires n to take on negative values over half its range. This is simply a matter of formalism, as we can see by looking at (10), which is exactly equivalent to (12). In view of (9), we see that

$$q_n = c_n e^{-j\phi_n}$$

$$h_n = c_n e^{j\phi_n}$$

Then we can write the bracketed term under the summation sign in (10) as

$$\frac{1}{2} c_n \left\{ e^{j(n\vartheta - \phi_n)} + e^{-j(n\vartheta - \phi_n)} \right\} = c_n \cos(n\vartheta - \phi_n)$$

Substitution of this result in (10) would give us (8). In the notation of (12), each Fourier component is the sum of the terms in n and -n. Each component can be thought of as the sum of two oppositely rotating vectors, with the sign of n determining the sense of rotation.

Finally, we note two essential properties of the Fourier representation. First, it is unique: identical functions have identical representations, and vice versa. Second, it is complete except at discrete points: except at discontinuities, the representation contains all of the information that is present in the function itself, and vice versa.

III. Fourier Transforms and the Fourier Integral

The Fourier series representation can be generalized to include non-periodic functions. This is achieved by taking limits as the period tends to infinity. The principal change in the representation is that in this case we have a continuous distribution of frequencies instead of a discrete series of harmonics. Therefore the series is replaced by an integral. We have

$$f(\vartheta) = \int_{-\infty}^{\infty} g(\omega)\, e^{j\omega\vartheta}\, d\vartheta \tag{14}$$

$$g(\omega) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(\vartheta)\, e^{-j\omega\vartheta}\, d\vartheta \tag{15}$$

These relations are analogous to (12) and (13). They comprise a <u>Fourier transform pair</u>. We shall speak of (15) as the direct transform, and of (14) as the inverse transform. We can think of $g(\omega)$ as a spectral function describing $f(\vartheta)$.

From (14) and (15) we obtain directly the Fourier integral representation of $f(\vartheta)$:

$$f(\vartheta) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(\vartheta')\, e^{j\omega(\vartheta-\vartheta')}\, d\vartheta'\, d\omega \tag{16}$$

This is analogous to the Fourier series representation of a periodic function.

The conditions a function must meet in order to have a Fourier integral representation are analogous to those for a periodic function to be expressible as a Fourier series. It must have no more than a finite number of discontinuities in any finite interval, and it must be absolutely integrable from $-\infty$ to $+\infty$:

$$\int_{-\infty}^{\infty} \left| f(\vartheta) \right| d\vartheta$$

must exist. Similarly, the representation possesses the same features of uniqueness and completeness as before.

In two dimensions, the transform relations are

$$f(x,y) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(u,v)\, e^{j(ux+vy)}\, du\, dv \tag{17}$$

$$g(u,v) = \frac{1}{4\pi^2}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, e^{-j(ux+vy)}\, dx\, dy \tag{18}$$

To express these in polar coordinates, we define the radial variables

$$r = \sqrt{x^2 + y^2}$$

$$\rho = \sqrt{u^2 + v^2}$$

and the angular variables

$$\vartheta = \tan^{-1} y/x$$

$$\varphi = \tan^{-1} v/u$$

Then

$$f(r,\vartheta) = \int_0^\infty \int_0^{2\pi} g(\rho,\varphi)\, e^{jr\rho\cos(\varphi-\vartheta)}\, \rho\, d\varphi\, d\rho \tag{19}$$

$$g(\rho,\varphi) = \frac{1}{4\pi^2} \int_0^\infty \int_0^{2\pi} f(r,\vartheta)\, e^{-jr\rho\cos(\varphi-\vartheta)}\, r\, d\vartheta\, dr \tag{20}$$

IV. Some Properties of Fourier Transforms

Fourier transforms have a number of properties which frequently are of use in practice. Let us define the operators

$\mathfrak{F}$ = direct Fourier transform

$\mathfrak{F}^{-1}$ = inverse Fourier transform

Then we can write a number of corresponding pairs of properties. Each pair is symmetrical except for occasional changes of sign.

$$\left.\begin{aligned} \mathfrak{F}\left[a_1 f_1(\vartheta) + a_2 f_2(\vartheta)\right] &= a_1 g_1(\omega) + a_2 g_2(\omega) \\ \mathfrak{F}^{-1}\left[a_1 g_1(\omega) + a_2 g_2(\omega)\right] &= a_1 f_1(\vartheta) + a_2 f_2(\vartheta) \end{aligned}\right\} \tag{21}$$

$$\left.\begin{aligned} \mathfrak{F}\left[\frac{d^n f(\vartheta)}{d\vartheta^n}\right] &= (j\omega)^n g(\omega) \\ \mathfrak{F}^{-1}\left[\frac{d^n g(\omega)}{d\omega^n}\right] &= (-j\vartheta)^n f(\vartheta) \end{aligned}\right\} \quad (22)$$

$$\left.\begin{aligned} \mathfrak{F}\left[\int_{-\infty}^{\vartheta} f(\vartheta')\,d\vartheta'\right] &= (j\omega)^{-1} g(\omega) \\ \mathfrak{F}^{-1}\left[\int_{-\infty}^{\omega} g(\omega')\,d\omega'\right] &= (-j\vartheta)^{-1} f(\vartheta) \end{aligned}\right\} \quad (23)$$

$$\left.\begin{aligned} \mathfrak{F}\left[\vartheta^n f(\vartheta)\right] &= (j)^n \frac{d^n g(\omega)}{d\omega^n} \\ \mathfrak{F}^{-1}\left[\omega^n g(\omega)\right] &= (-j)^n \frac{d^n f(\vartheta)}{d\vartheta^n} \end{aligned}\right\} \quad (24)$$

$$\left.\begin{aligned} \mathfrak{F}\left[f_1(\vartheta) f_2(\vartheta)\right] &= \int_{-\infty}^{\infty} g_1(\omega') g_2(\omega-\omega')\,d\omega' = \int_{-\infty}^{\infty} g_1(\omega-\omega') g_2(\omega')\,d\omega' \\ \mathfrak{F}^{-1}\left[g_1(\omega) g_2(\omega)\right] &= \frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(\vartheta') f_2(\vartheta-\vartheta')\,d\vartheta' = \frac{1}{2\pi}\int_{-\infty}^{\infty} f_1(\vartheta-\vartheta') f_2(\vartheta')\,d\vartheta' \end{aligned}\right\} \quad (25)$$

$$\left.\begin{aligned} \mathfrak{F}\left[\int_{-\infty}^{\infty} f_1(\vartheta') f_2(\vartheta-\vartheta')\,d\vartheta'\right] &= \mathfrak{F}\left[\int_{-\infty}^{\infty} f_1(\vartheta-\vartheta') f_2(\vartheta')\,d\vartheta'\right] = 2\pi\, g_1(\omega) g_2(\omega) \\ \mathfrak{F}^{-1}\left[\int_{-\infty}^{\infty} g_1(\omega') g_2(\omega-\omega')\,d\omega'\right] &= \mathfrak{F}^{-1}\left[\int_{-\infty}^{\infty} g_1(\omega-\omega') g_2(\omega')\,d\omega'\right] = f_1(\vartheta) f_2(\vartheta) \end{aligned}\right\} \quad (26)$$

$$\left.\begin{aligned} \mathfrak{F}\left[e^{j\omega'\vartheta} f(\vartheta)\right] &= g(\omega-\omega') \\ \mathfrak{F}^{-1}\left[e^{j\omega\vartheta'} g(\omega)\right] &= f(\vartheta+\vartheta') \end{aligned}\right\} \quad (27)$$

$$\left.\begin{aligned} \mathfrak{F}\left[f(\vartheta-\vartheta')\right] &= e^{-j\omega\vartheta'} g(\omega) \\ \mathfrak{F}^{-1}\left[g(\omega-\omega')\right] &= e^{j\omega'\vartheta} f(\vartheta) \end{aligned}\right\} \quad (28)$$