INTEGRATION OF A DC SIGNAL MIXED WITH NOISE FILTERING AND USEFUL RATE OF SAMPLING

BY MARC VINOKUR

NATIONAL RADIO ASTRONOMY OBSERVATORY

NOVEMBER 1961

INTEGRATION OF A DC SIGNAL MIXED WITH NOISE --
FILTERING AND USEFUL RATE OF SAMPLING

Let us consider a receiver fed by a constant level signal. Let $S(t) = S$ be the corresponding DC component, after detection, with which is mixed the receiver noise $X_0(t)$. Its power density spectrum is $A_0(\nu)$, its correlation function:

$$\rho_0(\tau) = \int_0^\infty A_0(\nu) \cos 2\pi\nu\tau \, d\nu$$

and its rms:

$$\sigma_0 = \sqrt{\rho_0(0)}$$

After passing through a filter having a power frequency response $g_1^2(\nu)$, such that $g_1^2(0) = 1$ in order not to modify the DC signal, the noise added to S becomes $x_1(t)$, with a power density spectrum of:

$$A_1(\nu) = A_0(\nu) . g_1^2(\nu). \quad (1)$$

(Theorem of Bochner Khintchine)

Let $\rho_1(\tau)$ be its correlation function, and σ_1 its rms. By a continuous integration during a time T (i.e., $\frac{1}{T} \int_0^T [S(t) + X_1(t)] \, dt$) one reduces the rms of the error on S by a factor $k\sqrt{T}$ ($k = 1$ when T is big enough compared to the time constant of the filter):

$$\sigma_2 = \frac{\sigma_1}{\sqrt{T}}$$

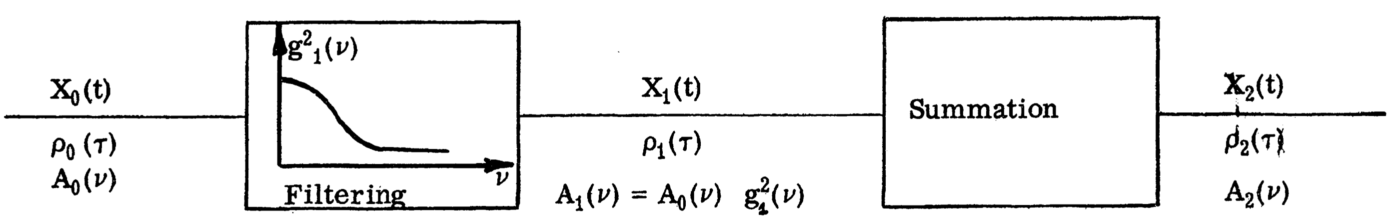

FIGURE 1

Actually, one knows that a continuous integration is superfluous since the noise has a certain statistical memory due to the limitation of its spectrum by the filtering, in other words since the values of this random function are not entirely independent. Consequently, the summation of the values of only a sufficient discrete number of points must reduce the rms of the error on S to a value σ_2' very little different from σ_2 or even equal to σ_2 in the ideal case of a strictly limited bandwidth.

A usual approach to this problem is to apply to the noise record the result of the sampling theorem for a strictly band limited function: A function f(t) [whose Fourier transform is zero for all frequencies greater than B] can be identically represented by:

$$f(t) \sum_{k=-\infty}^{+\infty} f\left(\frac{k}{2B}\right) \frac{\sin. \pi\ (2Bt-k)}{\pi\ (2Bt-k).}$$

Therefore, f(t) is entirely determined by the knowledge of its values for points at intervals $\frac{1}{2B}$ -- if T is the time of observation, n = 2BT+1 points are sufficient to restore f(t).

Actually, there are two objections: First, the problem is not to know a particular noise function, but only to reduce by summation the statistical fluctuation which affects S to its theoretical minimum σ_2. A second objection lies in the fact that it has been necessary to assume that the bandwidth of the filter is strictly limited.

The purpose of the following is to calculate directly the rms σ_2' of the noise which affects the signal S, after summation of values sampled at an arbitrary rate $1/\Theta$, as a function of Θ, of the power frequency response of the filter, and of the time of integration T. It will be then possible to define the useful rate of sampling by the value Θ_0 of Θ for which σ_2' is only greater by an acceptable factor of p cent to the theoretical minimum value σ_2, corresponding to $1/\Theta \rightarrow \infty$. By sampling S(t) +X(t) at intervals Θ, one obtains after summation of n values:

$$nS + \sum_{k=0}^{n-1} X_1(t_0 + k\Theta).$$

where t_0 corresponds to the beginning of the integration. Therefore, to the signal S is added the noise:

$$X_2(t_0) = \frac{1}{n} \sum_{k=0}^{n-1} X_1(t_0 + k\Theta).$$

Its correlation function is:

$$\rho_2(\tau) = \overline{X_2(t)\ \ X_2(t-\tau)} = \frac{1}{n^2} \sum_{k'=0}^{n-1} \sum_{k''=0}^{n-1} \overline{X_1(t + k'\Theta).\ X_1(t + k''\Theta - \tau)}$$

From:

$$\rho_1(\tau) = \overline{X_1(t).\ X_1(t-\tau)} = \int_0^{\infty} A_1(\nu) \cos 2\pi\nu\tau d\nu$$

one has successively:

$$\rho_2(\tau) = \frac{1}{n^2} \left\{ n\rho_1(t) + \sum_{k=1}^{n-1} (n-k) \quad [\rho_1(k\Theta + \tau) + \rho_1(k\Theta - \tau)] \right\}$$

$$= \frac{1}{n^2} \int_0^{\infty} R \left\{ n e^{2\pi j\nu\tau} + \sum_{k=1}^{n-1} (n-k) \quad [e^{2\pi j\nu(k\Theta+\tau)} + e^{2\pi j\nu(k\Theta+\tau)}] \right\} A_1(\nu)\, d\nu$$

$R\{\ \}$ = real part of $\{---\}$

And finally:

$$\rho_2(t) = \int_0^\infty A_1(\nu) \frac{\sin.^2 n\pi\nu\Theta}{n^2 \sin^2\pi\nu\Theta} \cos 2\pi\nu\tau d\nu$$

After summation, the power density spectrum of $X_2(t)$ is then:

$$A_2(\nu) = A_1(\nu) \frac{\sin.^2 n\pi\nu\Theta}{n^2 \sin.\ \pi\nu\Theta}$$

Therefore, the summation of n values of the noise at intervals Θ acts as a linear filter having a power frequency response

$$\boxed{g^2_2(\nu) = \frac{\sin.^2 n\pi\nu\Theta}{n^2 \sin.^2 \pi\nu\Theta}}$$

This is a comb filter, the lobes of which are spaced at intervals $1/\Theta$. The total area of a lobe with its secondaries is equal to $1/n\Theta = 1/T$; the width between first nulls is $\frac{2}{n\Theta} = \frac{2}{T}$.

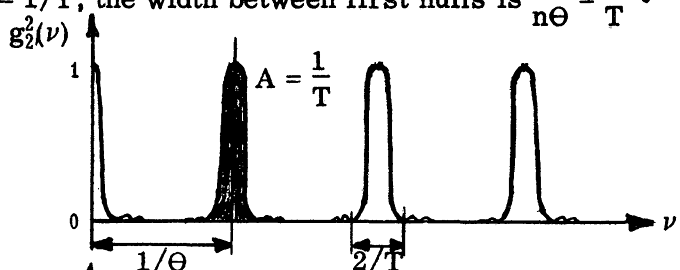

FIGURE 2

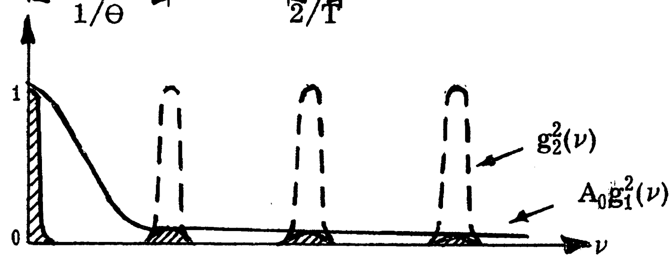

FIGURE 3

After filtering and summation, the power density spectrum of the noise is:

$$A_2(\nu) = A_1(\nu).\ g_2{}^2(\nu) = A_0(\nu).\ g^2{}_1(\nu).\ g_2{}^2(\nu)$$

with $A_0(\nu) \doteqdot A_0$ = constant

Therefore

$$\sigma'^2_2 = \rho_2(0) = \int_0^\infty A_2(\nu) d\nu$$

is represented by the hatched area on figure 3.

One sees that this hatched area comprises two parts:

(a) The first half-lobe which cannot be avoided since the filter must let pass the DC signal, its area is $\frac{1}{2}\ \frac{A_0}{T}$ and represents the minimum variance $\sigma_2{}^2$

(b) All of the other lobes, distant from each other by $1/\Theta$; their total area is a function of $g^2{}_1(\nu)$ and of $1/\Theta$, and decreases as $1/\Theta$ increases, tending to zero when $1/\Theta \longrightarrow \infty$ (the case of continuous integration).

It is therefore possible to define the useful rate of sampling $1/\Theta$, by the value Θ_0 of Θ, for which this remaining area can be considered as small enough compared to $\frac{1}{2}\,\frac{A_0}{T}$.

Example: Useful rate of sampling in the case of an RC filter -

With RC = τ, the power frequency response of an RC filter is:

$$g^2{}_1(\nu) = \frac{1}{1 + (2\pi\nu\tau)^2}$$

For: $1/\Theta = 1/2\tau$, one finds:

$$\frac{\sigma'_2 - \sigma_2}{\sigma_2} < 15\%$$

For: $1/\Theta = 1/\tau$:

$$\frac{\sigma'_2 - \sigma_2}{\sigma_2} < 3\% \qquad \text{(fig. 4)}$$

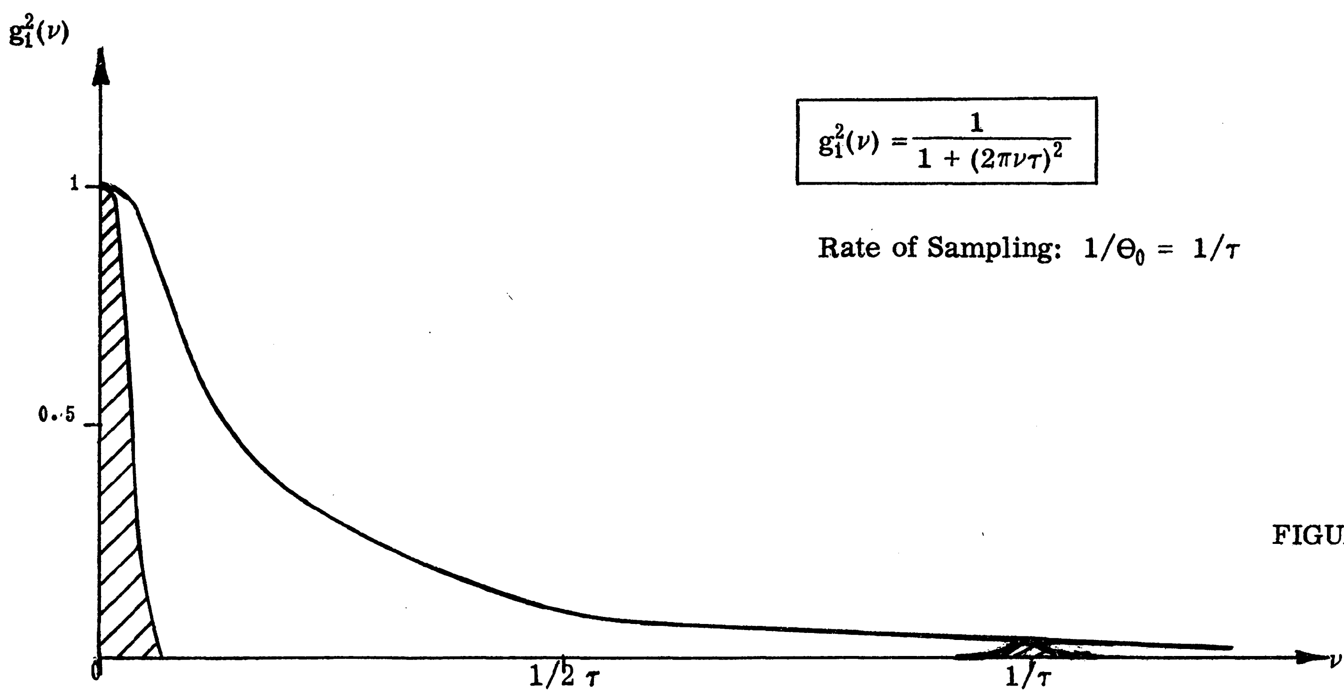

FIGURE 4

For: $1/\Theta = 2/\tau$:

$$\frac{\sigma'_2 - \sigma_2}{\sigma_2} < 0.5\%$$

One can consider a rate of sampling of $1/\Theta = 1/\tau$ as a reasonable value.

Remark

In the case of a strictly limited bandwidth B, the useful rate of sampling is exactly: $1/\Theta = B + \frac{1}{T}$,

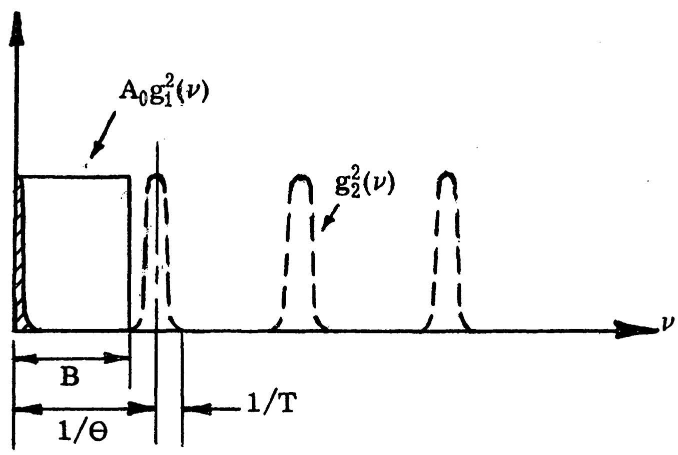

FIGURE 5

is exactly: $1/\Theta = B + \frac{1}{T}$.

This means that, in order to reduce the variance to its minimum, the number of points which must be sampled in a time of observation T is:

$$\boxed{n = BT + 2}$$

If the problem had been to reconstruct the record of noise, the number of points necessary for this restitution would have been:

$$n = 2BT + 1$$

thus twice as much.

Reference

A. Angot - Compléments de Mathématiques. Ed. Revue d' Optique. 1952. 644.

NATIONAL RADIO ASTRONOMY OBSERVATORY
Green Bank, West Virginia

Date: September 12, 1963

From: M. Vinokur and N. J. Keen

Subject: Numerical Calculations for the Report "Integration of a DC Signal Mixed With Noise: Filtering and Useful Rate of Sampling" by Marc Vinokur (NRAO, November 1961)

On page 4 of the original paper

$$\frac{\sigma' - \sigma}{\sigma} = \epsilon \text{ (rms error)} \qquad (1)$$

The mean square error is given by

$$R = \frac{\sigma'^2 - \sigma^2}{\sigma^2} \qquad (2)$$

From (1)

$$\sigma' = \sigma(1 + \epsilon)$$

From (2)

$$R = \frac{\sigma^2(1 + \epsilon)^2 - \sigma^2}{\sigma^2}$$

$$= 2\epsilon + \epsilon^2$$

And for $\epsilon << 2$

$$R \simeq 2\epsilon$$

Since the area under the RC filter curve in Figure 4 is given by

$$g_1^2(\nu) = \frac{1}{1 + (2\pi\nu\tau)^2}$$

then considering only the central cross-hatched area gives the mean square error

$$R = 2\sum_{n=1}^{\infty} \frac{1}{1 + \left(\frac{2\pi n\tau}{\Theta}\right)^2} \qquad (3)$$

where Θ = sampling interval

and τ = RC time constant.

Although (3) may not be summed to infinity by analytical means, the value

$$R_{500} = 2\sum_{n=1}^{500} \frac{1}{1 + \left(\frac{2\pi n\tau}{\Theta}\right)^2}$$

is taken. It may be shown that $R - R_{500}$ is negligible, by comparing with series which may be summed to infinity. Hence the digital computer gives

τ	R_{500}	ϵ
Θ	8%	4%
2Θ	2%	1%