ON THE PHASE CENTER OF A PARABOLOID

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If one goes a very long ways from a paraboloid radiating a coherent oscillator, and investigates the phase of the radiation pattern, one may trace out surfaces of constant phase. For regions sufficiently near the center of the main beam, these surfaces may be approximated by a sphereoid. If this sphereoid is a sphere, we may say that the paraboloid has a phase center, located at the center of the sphere.

In the classical, ray tracing, approach, the rays leaving a point fed parabola are parallel, so that the phase center is located at $-\infty$. This is not actually the case. The phase center is calculated below for a point fed parabola for both a scalar field (isotropic feed) and for the electromagnetic field (dipole feed). The process of setting up the equations is simplified if we consider a distant radiator and a receptive feed instead of the opposite. Let us first consider the case of a scalar field.

Let us define Cartesian (x,y,z) and cylindrical (r,ϕ,z) coordinates centered on the vertex of the paraboloid, shown in fig. 1. The equation of the paraboloid is

$$z = \frac{r^2}{4f} \tag{1}$$

Let us consider the response of the paraboloid to a source at a large distance, $\underline{L}$. Without loss of generality, we may consider the source to lie in the x-z plane. Let the angle between the direction to the source and the z axis be θ. If L is very large compared to all other dimensions, the field in the vicinity of the paraboloid is, to sufficient accuracy

$$A = A_o\, e^{ikL}\, e^{-ik\left(\frac{r^2}{4f}\cos\theta + r\cos\phi\sin\theta\right)} \qquad (2)$$

where $k = \frac{2\pi}{\lambda}$ and the phase was taken to be zero at the source. The incident field on the paraboloid is found by substituting (1) into (2)

$$A_p = A_o\, e^{ikL}\, e^{-ik\left(\frac{r^2}{4f}\cos\theta + r\cos\phi\sin\theta\right)}$$

The field is reflected with a change of phase of π. We may now use Huygen's principle to find the contribution of this point to the focus field. The amplitude must be diminished by $\frac{1}{\ell}$, where ℓ is the distance to the focus, and the phase is changed by $e^{ik\ell}$. Utilizing the property that ℓ + (dist. to directorix) = constant

$$\ell = f + \frac{r^2}{4f} \qquad (3)$$

So the contribution to the field at the focus of an element located at (r,ϕ) on the paraboloid is

$$A_f = -\frac{1}{f + \frac{r^2}{4f}}\, A_o\, e^{ik\left[\frac{r^2}{4f}(1-\cos\theta) - r\cos\phi\sin\theta\right]}\, e^{ik(L+f)} \qquad (4)$$

with uniform illumination, the field seen by the detector is

$$A_d = \int_0^R \cos\theta\, r dr \int_0^{2\pi} d\phi\, A_f \qquad (5)$$

A wave with a phase center near the origin of coordinates is represented for small θ by a phase varying as a quadratic polynomial in θ. Therefore, let us discard terms containing powers of θ higher than θ^2.

$$A_d = -A_o \cos\theta \, e^{ik(L+f)} \int_0^R \int_0^{2\pi} \frac{1}{f + \frac{r^2}{4f}} e^{ik\left[\frac{r^2}{4f}(1-\cos\theta) - r\cos\phi\sin\theta\right]} d\phi r dr$$

$$\approx -A_o \cos\theta \, e^{ik(L+f)} \int_0^R \int_0^{2\pi} \frac{1}{f + \frac{r^2}{4f}} \left(1 - ikr\theta\cos\phi + \frac{ikr^2}{8f}\theta^2 - \frac{r^2k^2}{2}\theta^2\cos^2\phi\right)$$

$$d\phi r dr \qquad (6)$$

$$= -2\pi A_o \cos\theta \, e^{ik(L+f)} \int_0^R \frac{1}{f + \frac{r^2}{4f}} \left(1 + \frac{ikr^2}{8f}\theta^2 - \frac{r^2k^2}{4}\theta^2\right) r dr$$

The phase angle of the integral is given by the imaginary part divided by the real part.

$$\Delta\phi \approx \frac{\int_0^R \frac{1}{f + \frac{r^2}{4f}} \frac{kr^2}{8f} \theta^2 r dr}{\int_0^R \frac{1}{f + \frac{r^2}{4f}} r dr} \qquad (7)$$

$$= \frac{1}{2}\theta^2 kf \, g\left(\frac{R}{2f}\right)$$

where

$$g(x) = \frac{x^2}{\log_e(1 + x^2)} - 1 \tag{8}$$

For 2R = 85' and f = 36', g = .166, fg = 6!0.

The phase is $\phi = k(L + f + \frac{1}{2}\theta^2 fg)$, so a path of constant phase is $L(\theta) = L_o - \frac{1}{2}\theta^2 fg$, which has a radius of curvature $L_o - fg$ which means that the phase center is at r = 0, z = fg.

It is now interesting to consider if the vector nature of the electromagnetic field has any effect on this argument. Let us consider a paraboloid fed with a point dipole, and retrace the above argument with a vector field

$$\vec{E} = \vec{E}_o e^{ikL} e^{-ik(z\cos\theta + x\sin\theta)} \tag{9}$$

The field on the paraboloid is

$$\vec{E}_P = \vec{E}_o e^{ikL} e^{-ik\left(\frac{r^2}{4f}\cos\theta + r\cos\phi\sin\theta\right)}$$

After reflection, the tangential component of $\vec{E}$ is reversed in phase, but the normal component is unchanged. The normal to the surface in x,y,z coordinates is

$$\vec{n} = \frac{1}{\sqrt{4f^2 + r^2}}(-r\cos\phi, -r\sin\phi, 2f) \tag{10}$$

and the normal component of $\vec{E}$ is $\vec{n}\,\vec{n}\cdot\vec{E}$, so the reflected field is

$$\vec{E}_{PR} = (-\vec{E}_o + 2\vec{n}\vec{n}\cdot\vec{E}_o)\, e^{ik\left(\frac{r^2}{4f}\cos\theta + r\cos\phi\sin\theta\right)} e^{ikL} \tag{11}$$

We wish to know the contribution of this field to the focus field. Firstly, only the component of $\vec{E}_{PR}$ perpendicular to the path will propagate. The direction of the path is

$$\vec{u} = \frac{1}{\sqrt{f^2 + \frac{r^2}{2} + \frac{r^4}{16f^2}}} \left(-r \cos \phi, \ r \sin \phi, \ f - \frac{r^2}{4f}\right) \qquad (12)$$

and the normal component of $\vec{E}_{PR}$ is

$$\vec{E}_{PR} - \vec{u}\,\vec{u} \cdot \vec{E}_{PR}.$$

We again have equation 3 for the distance to the focus, so the contribution of a field element located at r, ϕ on the paraboloid to the focus field is

$$\vec{E}_f = \frac{1}{\left(f + \frac{r^2}{4f}\right)} \left(-\vec{E}_o + 2\,\vec{n}\,\vec{n} \cdot \vec{E}_o + \vec{u}\,\vec{u} \cdot \vec{E}_o - 2\,\vec{u}\,\vec{u} \cdot \vec{n}\,\vec{n} \cdot \vec{E}_o\right) e^{ik\left[\frac{r^2}{4f}(1-\cos\theta) - r\cos\phi\sin\theta\right]} e^{ik(L+f)}$$

and the total focus field

$$\vec{E}_{fT} = \int_0^R \cos\theta \, r dr \int_0^{2\pi} d\phi \, \vec{E}_f.$$

If the point dipole feed is linearly polarized in direction $\vec{p} = (p_1, p_2, 0)$, the electric field seen by the detector is

$$E_m = \vec{E}_{fT} \cdot \vec{p}$$

$$E_m = e^{ik(L+f)} \int_0^R \int_0^R \frac{\cos\theta}{\left(f + \frac{r^2}{4f}\right)} \left(-\vec{E}_0 \cdot \vec{p} + 2\,\vec{p} \cdot \vec{n}\,\vec{n} \cdot \vec{E}_0 + \vec{p} \cdot \vec{u}\,\vec{u} \cdot E_0 \right.$$

$$\left. - 2\,\vec{p} \cdot \vec{u}\,\vec{u} \cdot \vec{n}\,\vec{n} \cdot \vec{E}_0 \right) e^{ik\left[\frac{r^2}{4f}(1 - \cos\theta) - r\cos\phi\sin\theta\right]} \, d\phi\, dr$$

If we now write this equation in terms of the individual components of the vectors, recalling $\vec{E}_0 = (E_{ox}, E_{oy}, E_{ox}\tan\theta)$, and discard terms of higher order than θ^2,

$$E_m = \cos\theta\, e^{ik(L+f)} \int_0^R \int_0^R \left[\frac{1}{f + \frac{r^2}{4f}} \left(-E_{ox}\,p_1 - E_{oy}\,p_2 \right) \right.$$

$$+ 2\,\frac{1}{4f^2 + r^2} \left(-p_1\, r\cos\phi - p_2\, r\sin\phi \right) \left(-r\cos\phi\, E_{ox} \right.$$

$$\left. - r\sin\phi\, E_{oy} + 2f\,\theta\, E_{ox} \right) + \frac{1}{f^2 + \frac{r^2}{2} + \frac{r^4}{kf^2}} \left(-p_1\, r\cos\phi - p_2\, r\sin\phi \right)$$

$$\left(-r\cos\phi\, E_{ox} - r\sin\phi\, E_{oy} + \theta\, E_{ox} f - \frac{r^2}{4f}\,\theta\, E_{ox} \right)$$

$$- 2\,\frac{1}{4f^2 + r^2}\,\frac{1}{f^2 + \frac{r^2}{2} + \frac{r^4}{16f^2}} + \left(-p_1\, r\cos\phi - p_2\, r\sin\phi \right)$$

$$\left(\frac{r^2}{2} + 2f^2 \right) \left(-r\cos\phi\, E_{ox} - r\sin\phi\, E_{oy} \right.$$

$$\left. \left. + 2f\theta E_{ox} \right) \right] \left[1 + ik\,\frac{r^2}{8f}\,\theta^2 - r\cos\phi\,\theta - k^2 r^2 \cos^2\phi\,\theta^2 \right] d\phi\, dr$$

After algebraic manipulations in the first bracket, this expression simplifies to

$$E_m = \cos\theta \, e^{ik(L+f)} \int_0^R \int_0^{2\pi} \frac{4f}{4f^2 + r^2} \left[- E_{ox}\, p_1 - E_{oy}\, p_2 \right.$$

$$\left. + \frac{1}{4f^2 + r^2} (p_1\, r \cos\phi + p_2\, r \sin\phi)(r \cos\phi\, E_{ox} + r \sin\phi\, E_{oy}) \right]$$

$$\left[1 + 2k \left(\frac{r^2}{8f} \theta^2 - r \cos\phi\, \theta \right) - k^2 r^2 \cos^2\phi\, \theta^2 \right] d\phi\, dr$$

after performing the angular integral

$$E_m = 2\pi \cos\theta \, e^{ik(L+f)} \int_0^R \frac{4f}{4f^2 + r^2} \left[\left(E_{ox}\, p_1 + E_{oy}\, p_2 \right) \left(- 1 + \frac{r^2}{4f^2 + r^2} \right) \right.$$

$$+ \frac{ik\, r^2}{8f} \theta^2 \left(E_{ox}\, p_1 + E_{oy}\, p_2 \right) \left(-1 + \frac{r^2}{4f^2 + r^2} \right)$$

$$\left. - k^2 r^2 \theta^2 \left(E_{ox}\, p_1 + E_{oy}\, p_2 \right) \left(- \frac{1}{2} + \frac{\frac{3}{4} r^2}{4f^2 + r^2} \right) \right] r\, dr$$

The phase of the integral is again the ratio of the real and imaginary parts

$$\Delta\phi \approx \frac{\int_0^R \frac{1}{(4f^2 + r^2)^2} \frac{kr^2}{8f} dr}{\int_0^R \frac{1}{(4f^2 + r^2)^2} dr} \theta^2 = \frac{1}{2} \theta^2 kf\, h\left(\frac{R}{2f}\right)$$

where $h(x) = \left(\frac{1}{x^2} + 1\right) \log_e (1 + x^2) - 1$. For 2R = 85' and f = 36' h = .157, and fh = 5!6, and the phase center lies 5!6 in front of the vertex.

For the finite feeds, one must properly calculate the field in the vicinity of the focus, and dot this vector with the field due to the feed. However, an approximate answer might be given by multiplying the contribution from each area at (r, ϕ) by the gain of the feed over that of a point dipole in that direction. It should, however, be noted that this procedure does not give the right answer for the dipole feed, using the formula for an isotropic feed.

Knowing the location of the phase center, we can by geometry calculate the effect of a small pointing error θ on the phase of an interference pattern. It will be represented by terms of the order $L\theta^2$, $L\theta\Delta\phi$, $L\theta\ell$, and $L\theta\Delta h$ where L is a typical distance on the telescope mount, $\Delta\phi$ is the error in perpendicularity of the axes, Δh is the colimation error and ℓ is the distance of the phase center from the perpendicular line between the two axes. These terms are all probably smaller than those introduced by the deformation of the feed structure, and of the dish surface.

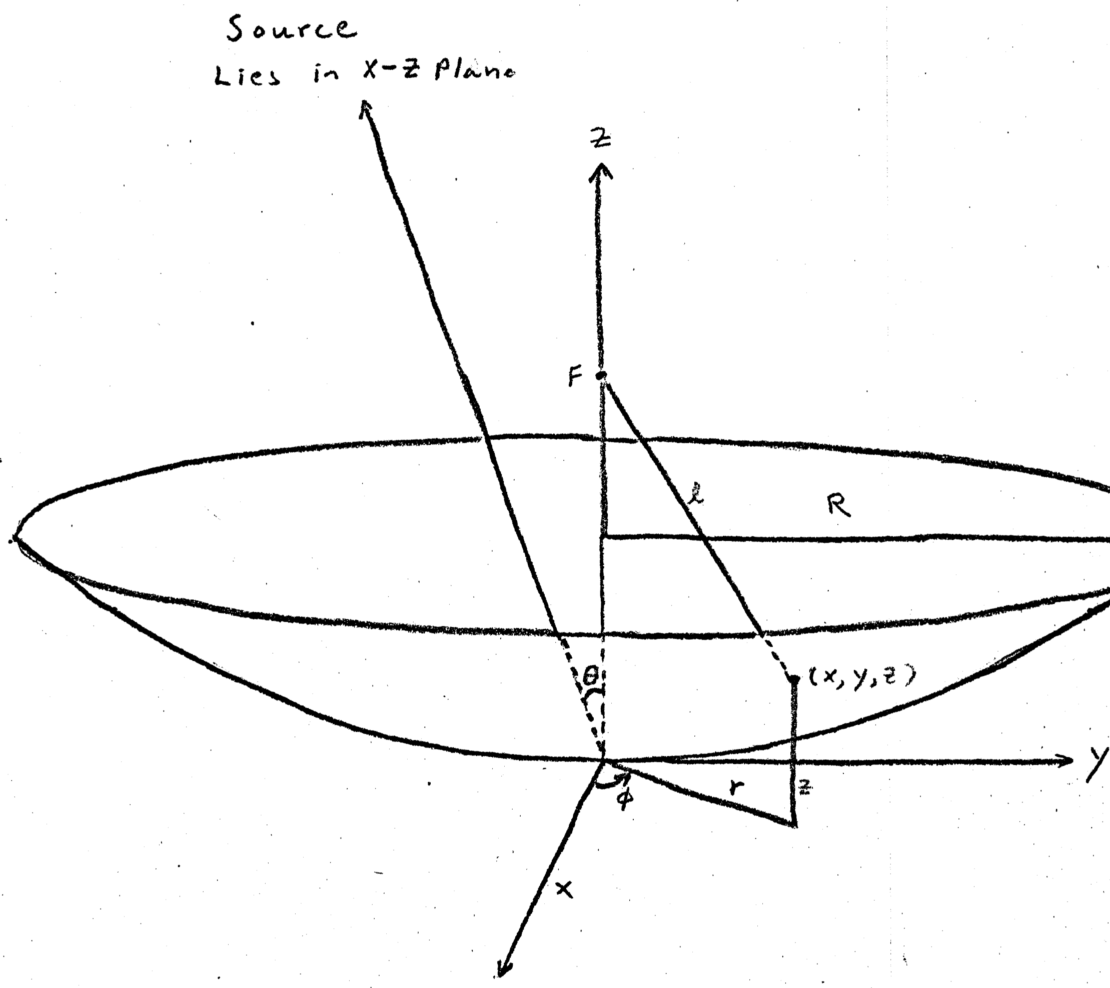

Figure 1