mmA Antenna Memo No. 22

VS/GR BK/

DAMPING AND VIBRATION CONTROL

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Abstract

One of the requirement for the proposed MMA antenna is the ability to switch rapidly between positions on the sky. This implies large acceleration of the structure and the excitation of various modes of oscillation in the structure is a major concern for any antenna structure. This memo is a brief review of existing damping mechanisms and vibration control methods for large structures. The information on this memo comes from many disciplines: a) structural vibration; b) damping materials; c) vibration control; d) smart material; e) smart structures and f) optimum control. The damping mechanisms existing include material damping, friction damping, viscoelastic damping, viscous damping, dynamic absorber, tuned mass damping(including rubber spring and magnetic tuned mass damping), piezoelectric damping and proofmass damping. Damping theory and treatment was developed earlier than the 1950's, but practical applications have been very limited until recently. A relatively new field in this area is known as smart structures, a technique involving built in sensors and actuators.

1.0 Introduction.

In the structure field, elasticity has been the basis of both static and dynamic analysis. Without damping, any energy applied to a structure, will be stored as strain energy, and released in kinetic form, this energy exchange producing structural vibration. If the frequency of excitation is at or near a natural frequency of the structure, resonance will occur. The excessive vibration will cause high stress, high displacement and will prevent the structure fulfilling the desired function. To reduce structural vibration levels several techniques may be employed. The first is to reduce the excitation at its source, the second is to alter(usually increase) the structural natural frequency, the third is by increasing structural damping. The first aspect is difficult to change if excitation is from the required function of the structure. Therefore in the past structural engineers have payed more attention to the second aspect since the performance involving damping is difficult to predict. However, increasing structural natural frequency requires high stiffness and may result in a heavier structure. With the development of vibration theory, material science, control method and computer software, increasing damping becomes more attractive in vibration suppression and control.

2.0 Passive and active damping technology.

The primary effects of damping in a structure are reduction of vibration amplitudes, with corresponding decreases in stresses, displacements and fatigue. Damping includes two classes: inherent damping and designed-in damping. Inherent damping includes material(or hysteretic) damping, damping in structural joints and environment caused damping such as aerodynamic damping. Material damping is caused by internal friction when the material is under deformation. Most structural materials have very low material damping. Table 1 lists the damping ratios of some materials. Damping in structural joints comes from the micro slippage on the joints. For modern structure, it is also small. Aerodynamic damping is even negligible. In summary, the level of inherent damping of a typical structure is usually less than 2 percent(Beards, 1983).

Designed-in damping is usually necessary when vibration suppression is required. Designed-in damping supplements inherent damping and can increase the structural damping by substantial and predictable amounts. Methods of designed-in damping can be classified as passive or active ones. Passive damping is simple and it involves no energy input and control system. Active damping is usually more complicated and it involves various kinds of sensors and actuators. Recently developed active damping with built-in sensors and actuators is called smart structure damping.

Summarizing all passive and active damping methods, designed-in damping for structures includes:

a) viscoelastic layered damping;
b) viscous device;
c) dynamic absorber;
d) tuned-mass damping:
   with rubber spring;
   with magnetic field;
e) piezoelectrics damping;
f) proofmass damping;
g) smart structure damping.

where a), b) c) and d) are passive damping; e) could be passive or active depending on the electrical circuit design; and f) and g) are active damping concept. The general trend of these damping methods are that simplicity decreases, control range and cost increase when a later method is adapted. The following paragraphs describe the mechanism of these damping methods.

3.0 Viscoelastic layered damping.

Viscoelastic materials(VEMs) are elastomeric materials whose long-chain molecules cause them to convert part of mechanical energy into heat when they are deformed. The stress and strain relationship of these materials can be represented in a complex field. The ratio of stress and strain with this kind of materials has two terms: one is a real term and the other is an imaginary term. The real term is called the storage modulus and the imaginary term is called the loss modulus. The expression is:

$$\sigma = (E' + iE'')\epsilon = E(1 + i\eta)\epsilon$$

where σ is stress; ε is strain; E' is the storage modulus; E" is the loss modulus and η is loss factor. The most important advantage of VEMs is their high loss factor. In other words, VEMs have higher material damping. However, the VEMs have low storage modulus, they are not suitable as structural supporting members. They only can be used as additional damping layers or as rubber like springs. In this section, only layered VEMs damping is discussed. There are several ways to apply VEM layers to structural members in order to increase structural damping. One way is to apply VEMs directly on the structural member surface. However during structural vibration only small fraction of total strain energy goes into the VEM layers, which makes the added damping being small. The most effective way is to apply the VEM layer between the structural member surface and another thin cover layer(Cheng, 1994). The cover layer is called constrained layer, which is made of the same material as the structural member and is fixed only at one end. The other end of the constrained layer is free, therefore the VEM layer undergoes a shear deformation when the member is in compression or extension. So the damping of the structure is increased greatly. This treatment is called constrained layer damping. For achieving high damping ratio and avoiding higher stress level in the VEM layers, successive layers of VEMs and cover metals can be applied on the structure members.

One restriction of using VEMs is that the storage modulus and loss factor of VEMs are both temperature and frequency dependent. Particularly, temperature has a greater effect on the storage modulus and loss factor of the VEMs. For a typical VEM, three distinct temperature regions are observed, namely the glassy, transition and rubbery regions. In the glassy region E is high and η is low. In the transition region E varies rapidly with temperature and η is high. In the rubbery region E varies more slowly with temperature and η is lower than in the transition region. At very high temperatures, irreversible thermal decomposition usually occurs. Figure 1 shows the variation of storage modulus and loss factor of VEM with temperature at fixed frequency and at low cyclic strain amplitude. The storage modulus of VEM can vary as many as five orders of magnitude over a narrow temperature range. VEMs have their applications in layered damping mostly in their transition regions, where high loss factors are observed. VEMs in rubbery region can be used for the tuned mass damper which is discussed in the later paragraph. In this region, both storage modulus and loss fact change very smoothly with temperature. Figure 2 is the graph of E and η versus frequency for

VEMs. Similarities exists between these two figures(Figures 1 and 2). If the effect of frequency and temperature on damping behavior are to be taken into account simultaneously, one most useful technique is called reduced frequency model(Torvik, 1980). The frequency in the above graph can be represented by a new term, called the reduced frequency $\alpha_T f$, where f is the frequency and α_T is a shift factor which is a function of temperature. High temperature corresponds a lower shift factor and low temperature corresponds a higher shift factor. Figure 3 is the figure of α_T for Polyisobutylene material.

Layered VEMs are efficient in vibration control, the frequency range in damping is very wide compared with other damping methods. By using layered VEMs, the damping coefficient can be as high as 0.5 to 1.3. However the design of layered VEM dampers has to consider the VEM properties in particular conditions. The design and manufacture of a large number of constrained layered structural members with different shapes and dimensions will be quite expensive. Therefore the most effective way in applying VEM layers is to select structural members which have higher stress level during structural vibration.

4.0 Viscous device.

Viscous devices dissipate energy via a true velocity dependent mechanism, typically by forcing a fluid through a precision orifice. The levels of loss obtainable by a viscous device are higher than those obtainable with VEM-layered structures. However viscous devices are limited in lower frequency range and the effective bandwidth is small. Their damping properties are also sensitive to change of temperature, though to a lesser degree than VEMs. This change is due to the viscosity of the fluid changing. Another disadvantage is that viscous devices are only most effective for axial deformations. If the vibrations are caused mainly by bending, the viscous device is difficult to apply.

5.0 Dynamic absorber.

Dynamic absorber is an important damping device, which generates inertia and reduces the vibration level of a protected structure. It is also called vibration absorber. In most cases, a dynamic absorber consists an additional mass and a spring. As in Figure 4, M, K, x are the mass, elastic coefficient and generalized coordinate of the protected construction; m, k, y are those for the absorber. For evaluating the efficiency of the absorber, parameters $\nu = m/M$, $f^2 = f_0^2/\omega_0^2$ play a very important role, where $\omega_0 = \sqrt{K/M}$ is the circular vibration frequency of the structure without an absorber, and $f_0 = \sqrt{k/m}$ is that of the dynamic absorber when x=0(without the effect of the protected system). ν is called relative mass and f^2 is called tuning of the absorber. The differential equations of the system vibrations are:

$$M\ddot{x}+Kx+k(x-y)=Qe^{ip_0 t}$$

$$m\ddot{y}+k(y-x)=0$$

The above equations have particular solution as:

$$x=X_0e^{ip_0t} \quad y=Y_0e^{ip_0t}$$

$$X_0=\frac{Q}{K}\frac{f^2-p^2}{\Delta} \quad Y_0=\frac{Q}{K}\frac{f^2}{\Delta}$$

$$\textit{where} \quad \Delta=(1-p^2)(f^2-p^2)-\nu p^2f^2 \quad p=p_0/\omega_0$$

When $\Delta=0$, the solution does not exist. In other words, the combined system now possesses two resonant frequencies $\omega_{1,2}$.

$$\omega_{1,2}=\sqrt{[[1+f^2(1+\nu)\pm\sqrt{[1+f^2(1+\nu)]^2-4f^2}]/2]}$$

However, when f=p, i.e. in the case of partial absorber frequency and forced frequency coincidence, it appears that $X_0=0$, $Y_0=-Q/k$. This is the case for which the protected construction remain static and the only displacement happens on the absorber. Therefore the dynamic absorber can be used to protect a structure at a particular frequency. In this case, the internal force in the absorber connection is equal to and fully balances the disturbing force. However, at other frequencies, the main mass will vibrate at various amplitudes. The amplitude frequency response for this system with and without the absorber is shown in Figure 5. The location of the two new resonance frequencies is determined totally by the relative mass ν as in Figure 6. For smaller relative mass of the absorber, the new resonant frequencies $\omega_{1,2}$ are very close each other, the damping bandwidth is small; increasing ν gives a better separation between the two frequencies and increases the damping bandwidth.

The design of a dynamic absorber is to adjust the values of ν and f so that the frequency bandwidth is acceptable, the amplitude of vibration of protected construction being very small. Dynamic absorber can only be used in limited system for a narrow frequency range as ν is related to the new resonant frequencies.

6.0 Tuned mass damper.

The application of dynamic absorber could be extended if the damping is added to the system. The damped absorbers are called tuned mass dampers. Such damper's sensitivity to the parameters' deviation from the optimum values is much lower than when damping is absent. This considerably increases their service reliability. At the same time, the added damping also reduces the amplitude of vibration even at the new resonant frequencies. Tuned mass damper is the simplest way to suppress structural vibration and is well suitable to the MMA antenna structures. Complete discussion of the tuned mass dampers is beyond the scope of this short memo. However, in this paragraph, some formulae of amplitude and frequency characteristics are provided.

The system of a tuned mass damper is similar to that shown in Figure 4, while the differential equations for the vibrations of the main mass and the damper mass are:

$$M\ddot{x}+Kx+k(x-y)+\mu_0(\dot{x}-\dot{y})=q(t)$$

$$m\ddot{y}+\mu_0(\dot{y}-\dot{x})+k(y-x)=0$$

$$where\quad q(t)=Q_0 p^{\alpha_1} e^{ip_0 t}$$

In the expression, α_1 gives the amplitude property of the excitation, For $\alpha_1=0$, the excitation has a constant amplitude and for $\alpha_1 = 2$, the amplitude is proportional to the frequency squared. The solution of the above equations are:

$$x=X_0 e^{ip_0 t} \qquad y=Y_0 e^{ip_0 t}$$

$$X_0=\frac{Q_0}{K}p^{\alpha_1}\frac{f^2-p^2+i\mu p}{b_1+i\mu p b_2} \qquad Y_0=\frac{Q_0}{K}p^{\alpha_1}\frac{f^2+i\mu p}{b_1+i\mu p b_2}$$

$$b_1=(1-p^2)(f^2-p^2)-\nu p^2 f^2 \quad b_2=1-p^2(1+\nu) \quad \mu=\mu_0/(m\omega_0)$$

From these expressions, it is possible to derive the frequency amplitude (dimensionless amplitude of displacement, $D=X_0K/Q_0$) diagram for the main structure. Figure 7 shows these relationships for different values of α, μ and f when $\nu=0.1$ (the damper mass is one tenth of the main mass). One can find that with damping introduced between main mass and damper mass, the response of the structure is much smoother. However, infinite damping will produce the situation as the undamped structure. In the diagram, there are two distinct intersection points of all the curves, these two points are called invariant points. The optimization of the tuned mass damper involves reducing the size of the peaks in the amplitude/frequency diagram of Fig. 7. This tuning process is critical in the tuned mass damper design. The optimization conditions of the system are listed in Table 2 for different values of α. In the table, P is the relative positions of the invariant points when the system is properly tuned, D is normalized amplitude of the protected structure. For $\alpha=0$, the sinusoid external force, $D^2=(2+\nu)/\nu$, the amplitude of the protected structure at the properly tuned condition is a function of only relative mass, smaller ν is, the effect of the damper is smaller too. In the table the value a, which is called sensitivity coefficient, is also listed. It represents the influence of the possible deviation of f^2 from f_{opt}^2

to the normalized amplitude. In the case of α=0, we have a=2.18 for ν=0.1, and a=7.07 for ν=0.01. Therefore smaller damper mass will require finer tuning of the system. Deviation of the damping from the optimum value has much less effect on vibration level of the protected structure. From the table, it can be found that the relative mass is a most important parameter of the tuned mass system. A relative mass of 0.02 of a fine tuned mass damper will bring normalized amplitude down to 10, which is equivalent to an additional 5 % damping. The influence from the damping within the main structure is straight forward. The added damping of the main structure will reduce the amplitude and will increase the damping effect. However, different from layered damping, tuned mass damper has to be placed at the points where vibration amplitudes are significant, while the layered damper has to be placed where strain energy is significant.

The damper mass is always smaller than the mass of protected structure. The VEMs can be used as spring component in tuned mass dampers since they have low storage modulus. Because the tuning here is critical for this application, the VEMs for the tuned mass damper are used in their rubbery region. In the rubbery region, the VEMs have stable storage modulus over a wide temperature range(this is important as heat will produced during structural damping). The loss factor of the material can act as the required damping of the system.

Rare earth magnets can be used in tuned mass damping system too. The magnets provide damping as metal moving between them. The eddy current produced within the metal surface generates heat and causes energy dissipation. To obtain the desirable level of damping, the variables that can be manipulated are the magnetic field strength, the volume of intersection of the field with the conductor, and the resistivity of the conductor. More precisely, the damping resulting from magnetic damping is:

$$c \sim (B^2 v)/\rho$$

where c = damping coefficient = force/ velocity; B= magnetic field strength; v= volume of field and conductor intersection and ρ= conductor resistivity. The resultant damping from this device is stable over a wider temperature range than the VEMs layered damper. It is suitable for structures working in harsh environments where neither VEMs nor viscous device are possible. Also the damping ratio can be adjusted easily. However, the damping coefficient from this device is usually less than that obtainable from viscous device. This may be not a disadvantage for the tuned mass damper, which requires no over damping.

7.0 Piezoelectric damper.

Piezoelectric ceramic materials have the unique ability to produce a strain when subjected to an electrical charge, and, conversely, they produce a charge when strained mechanically. A piezoelectric material can be used either passively or actively in structural damping. A passive usage of piezoelectric material is by shunting it with a passive electrical circuit, thereby turning vibrational strain energy into electrical energy that can be dissipated as heat energy by a resistor. Two types of circuits exist: a) a resistor alone. b) a resistor in series with an inductor. The maximum loss effect by resistor alone circuit is 0.425. Shunting with a resistor and inductor, along with the inherent capacitance of the piezoceramic, creates a resonant LRC circuit that is analogous to a mechanical tuned mass damper, except that it counters vibrational strain energy instead of kinetic energy. This can result in higher loss factor. Active control of the current in piezoelectric material will produce active damping of the structure. The principle of active control will be discussed in the following paragraph. From the cost viewpoint, the piezoelectric material may not suitable to large structures such as mmA antennas. So detailed discussion is not necessary.

8.0 Proofmass damping.

Proofmass damping has received attention recently because of its advantageous force-to-weight ratio(Clark and Robertshaw, 1989). Proofmass damping is, in fact, an tuned mass device with active control actuator. The passive tuned mass damping has the following weak points:

1) The effect of a tuned mass damping depends solely on the mass ratio ν, for smaller ν, the performance is limited;
2) When ν is small, the optimum damping required on the tuned mass is small too. Hence the amplitude frequency curves varies sharply around the system natural frequency. Small deviation from the optimum frequency will result in a significant loss of control performance.
3) The smaller the damping factor of tuned mass, the longer the time it takes until it reaches the steady state response condition. Hence at the beginning of excitation, it does not work effectively to suppress the response motion of a protected system.

All these weak points can be removed or reduced by introducing an active controlled actuator. The basic arrangement of a proofmass damper is a moving mass which is called proofmass and a linear actuator. Using the same notations as before, the equations are:

$$M\ddot{x}+K(x-z)+k(x-y)+\mu_0(\dot{x}-\dot{y})=q(t)-c(t)$$

$$m\ddot{y}+\mu_0(\dot{y}-\dot{x})+k(y-x)=c(t)$$

The optimum conditions of the proofmass damper can be derived by the procedure used for tuned mass damper. The active term of c(t) greatly changes the performance of the system. Nishimura(1992) gives a comparison between tuned mass and proofmass damping for the mass ratio $v=0.01$. For tuned mass damper, the frequency ratio $f=0.99$ and damping factor for the added mass is 0.061 make the maximum response(normalized amplitude) 14.2. By using a proofmass damper, the optimum frequency ratio $f=0.89$, the damping factor for the added mass is 0.274 and the maximum response is 3.00. The amplitude frequency curves of two conditions are in Figure 8. The improvement of the proofmass damper is significant.

The integration of the proofmass actuator into the system poses constraints on the control loop. This is because the position of the proofmass is limited with stops. Therefore the stroke/force applied is a nonlinear curve with saturation. In the design of proofmass damping, the proofmass should move within the limits and the system works within the linear region.

9.0 Smart structure damper.

A new development of vibration suppression may be found from the field of smart structure. An example using smart structure is the space station platform. Operation condition of these large space structures, e.g. targeting slew maneuvers and spacecraft docking, can induce unacceptable structural vibrations that take many minutes to naturally dampen out. Increased damping must be added to it. Mostly the damping is a combination of passive and active control systems. Generally, the smart structure involves sensors and actuators. The sensors used can be accelerometers, strain gauge, piezoelectric devices, and optical fiber apparatus. Especially the imbedded interferometric optical fiber strain sensors have been studied in detail. The actuators used for smart structures are piezoelectric material, shape memory alloys or polymers, intelligent structure or active members. The smart structure requires a sophisticated control system. Mostly they use state space optimum control technique. In a typical vibration control problem displacements can be defined as state variables. By proper modelling, state equations can be formed which include the control variables. In order to solve the required control variables, it may be necessary to minimize the energy of all the states and also the control variables. Detailed formulation of the optimum control theory can be found in related publications(Lewis, 1986, Sage, 1968).

10.0 Practical considerations of structural damping.

In the previous paragraphs, various methods of structure damping have been discussed. In particular, dynamic absorber and tuned mass damper are discussed in detail. In the discussion, a single mass spring system is assumed. This system has only one natural frequency and it is easily determined. However, the practical structure will be much more complicated than that. There are numerous modes and modal frequencies. Therefore, one damper can only work for one or few particular modes. However, the lowest modal frequencies are more important since they are easily excited by an outside force. For modal frequencies which are much higher than the excitation frequency, the vibration amplitude will be very much lower. The effect is negligible. When the damper is designed for a particular mode, the mass considered should be modal mass instead of structural mass. Mode mass is usually smaller. For ensuring the desirable performance, detailed modal analysis and frequency response analysis have to be performed during the design stage.

For fast switching excitation of the mmA antennas, the excitation bandwidth can be roughly estimated. If the switching distance is 3 degrees, for a settling accuracy of 1 arc second, the required accuracy is 0.01%. If the loop is a first order one, which is simple, it takes 9 time constants to converge to 0.01% accuracy. If the total settling time is 0.2 second, then the frequency band will up to 7.2 Hz(the total switching time includes slewing time, settling time and ring time, this memo is concerning only the structural ringing time). So we anticipate that the frequencies which are higher than 20 Hz will have little effect on the structure ringing. The offset structure of mmA has two lowest modes of about 8 Hz, and the rest of the modes all higher than 20 Hz. So damping on the two lowest modes(feed leg part) are necessary. These two lowest modes are in the directions parallel to the slant axis bearing plane. A shear tuned mass damping system may work for both modes. The panel surface deformation during switching is not too serious and only minor damping is adequate.

Considering these factors, the suggested damping for mmA antennas is: VEM layered damping at the panel adjusters; a rubber shear-type tuned mass damper for the feed leg structure, which works in both directions perpendicular to the slant axis and may be located on the back of the secondary mirror. By adding these damping devices, the dynamic performance of the antennas will improve significantly either satisfying the fast switching calibration requirements or reducing the time spent between observations. Comparably, the cost of structure damping is a small part of the total cost of the project.

Appendix: Relationship amongst various measures of damping

$$Q^{-1}=\eta=\Phi=\frac{E''}{E'}=2\zeta=\frac{\Delta W}{2\pi W}$$

Q	Quality factor
η	Loss factor
ϕ	Phase angle by which stress leads strain
E"	Loss modulus
E′	storage modulus

ξ damping ratio
ΔW energy loss per cycle
W maximum elastic stored energy

References

Beards, C. F., Structural vibration analysis: modelling analysis and damping of vibrating structures, Ellis Horwood Ltd. and John Wiley & Sons. New York, 1983.
Carman, G. P., Ed., Adaptive structures and material systems, The American society of mechanical engineers, AD Vol. 35, New York, 1993.
Cheng, J., Rapid position switching in radio telescope: structural damping using a constrained layer treatment, mmA memo 105, NRAO, 1994
Clark, W.W. and Robertshaw, H.H., A phase comparison of actuators for vibration control of flexible structures, Proc. of 30th structures, structural dynamics, and materials conference, Mobile, AL, 1989.
Hara, F., Damping-1988, American society of mechanical engineers, pvp. vol.133. 1988.
Hara, F, Active and passive damping - 1991, American society of mechanical engineers, PVP-vol. 211, 1991.
Haritos, G. K., Smart structures and materials, American society of mechanical engineers, ad vol. 24, 1991.
Kinra, V. K. and Wolfenden, A., Eds., M^3D: Mechanics and Mechanisms of Material Damping, STP 1169, ASTM, 1916 Race street, Philadelphia, 1992.
Korenev, B. G. and Reznikov, L. M., Dynamic vibration absorbers, John Wiley & Sons, New York, 1993.
Lewis, F.J., 1986, Optimal control, John Wiley & sons, New York.
Lindner, D.K., et al, Vibration suppression using a proofmass actuactor operating in stroke/force sateration, Journal of vibration and acoustics, Vol 113, 423-433, 1991.
Nishimura, I., et al. active tuned mass damper, Smart Mater. Struct. 1 306-311,1992.
Radcliffe, C.J., Active control of noise and vibration -1992, American society of mechanical engineers, DSC-vol 38, 1992.
Sage, A.P., 1968, Optimum systems control. Prentice Hall, inc., Englewood Cliffs, N.J.,
Sankar, T.S., Machinary dynamics - applications and vibration control problems, American society of mechanical engineers,DE vol. 18-2, 1989.
Torvik, P. J., Ed., Damping applications for vibration control, The American society of mechanical engineers, AMD Vol. 38, New York, 1980.
Walshaw, A. C., Mechanical vibration with applications, Ellis Horwood Ltd. and John Wiley & Sons. New York, 1984.

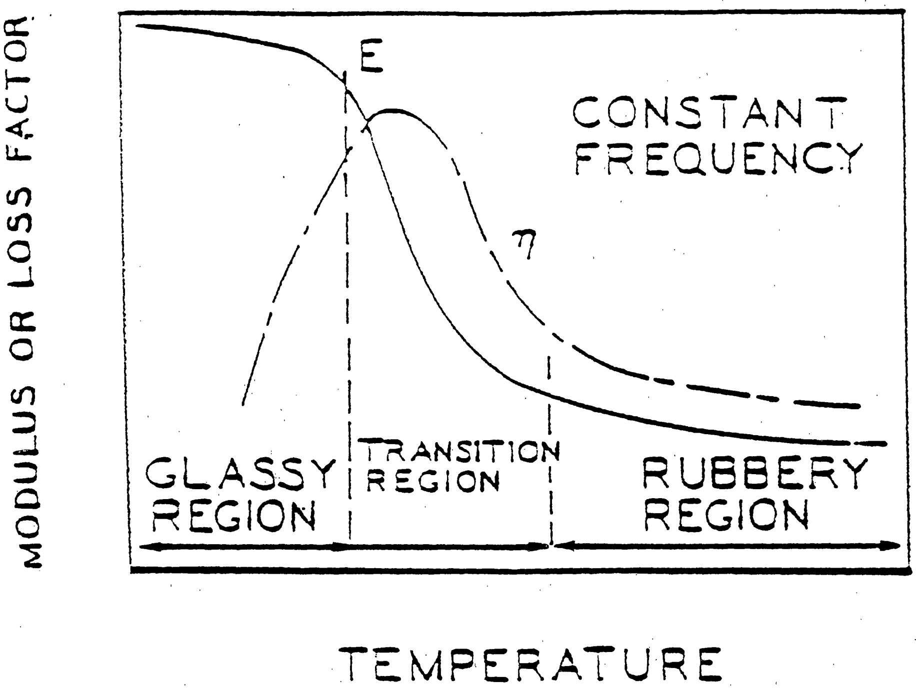

Figure 1 Variation of Young's modulus E and loss factor η with temperature for a viscoelastic material.

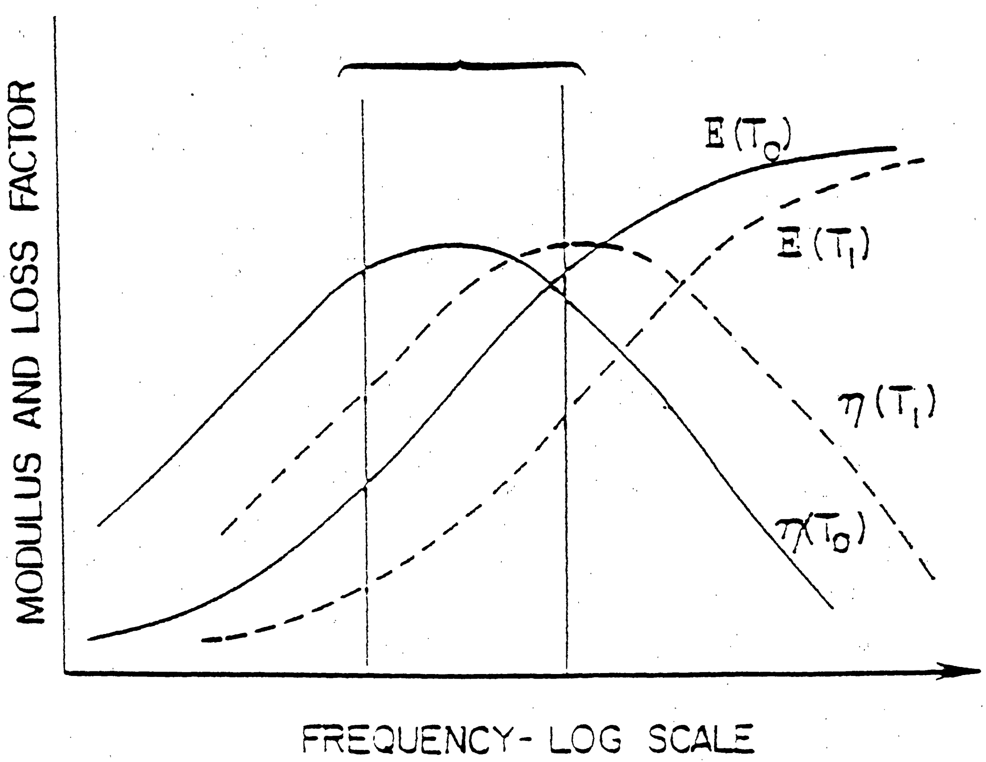

Figure 2 Variation of Young's modulus E and loss factor η with frequency for a vizcoelastic material.

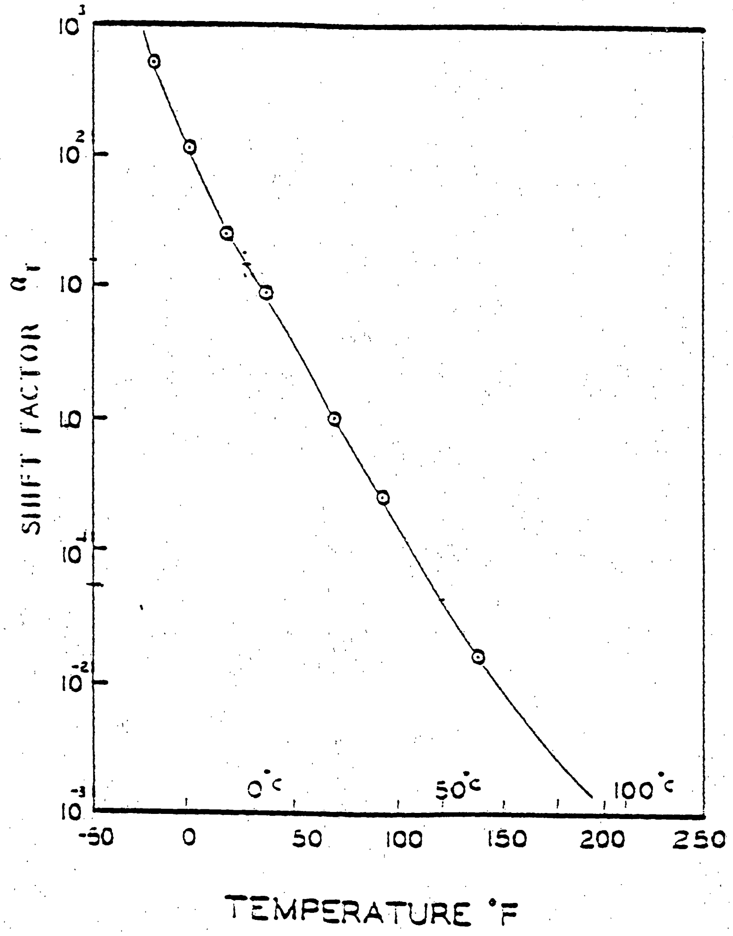

Figure 3 shift factor α versus temperature for Polyisobutylene material.

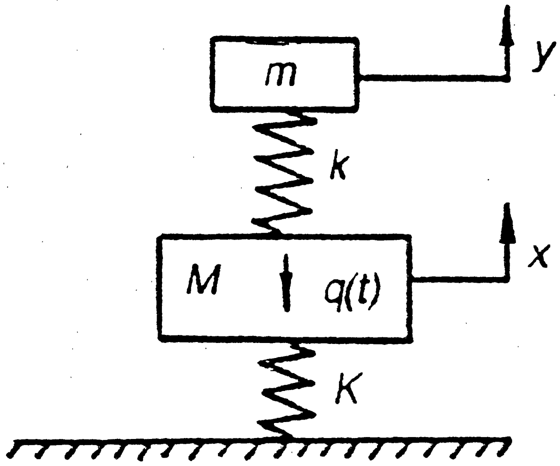

Figure 4 Diagram of a dynamic absorber with the structure protected.

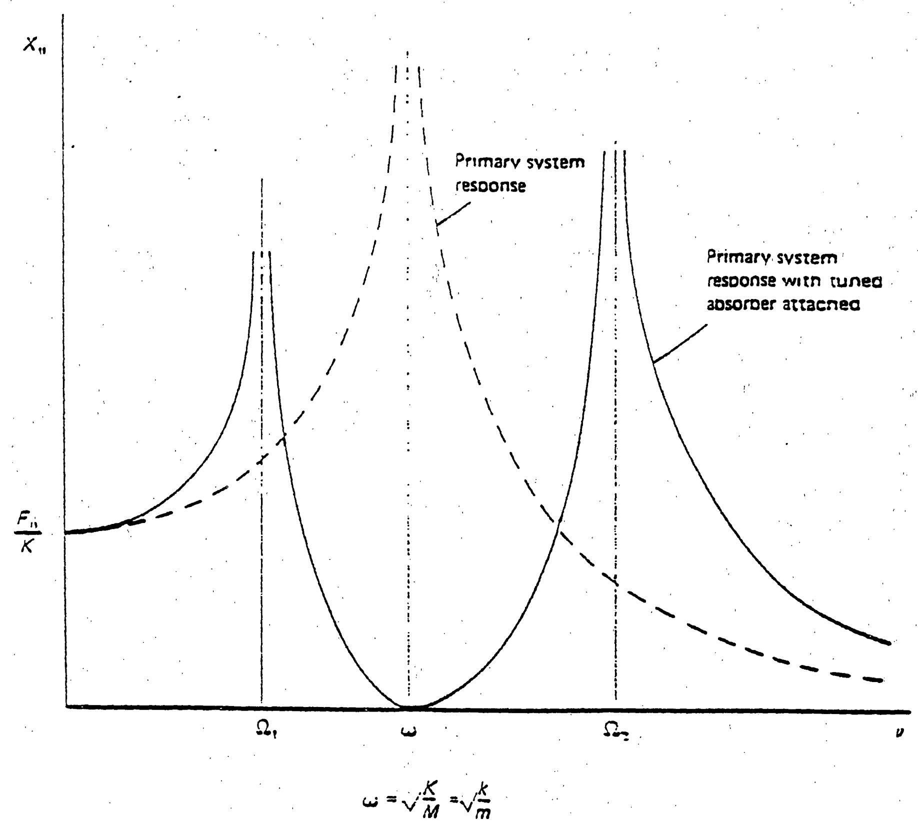

Figure 5 Amplitude frequency response for system with and without tuned absorber.

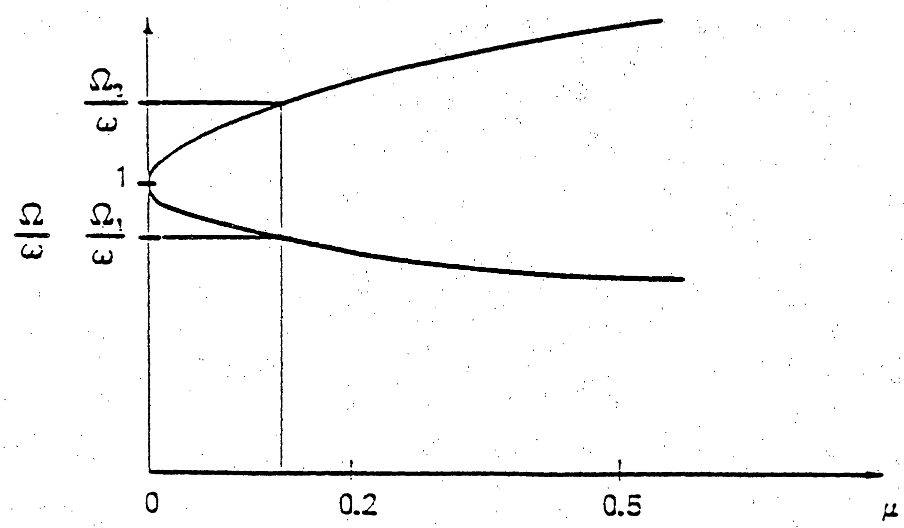

Figure 6 Effect of absorber mass ratio on natural frequency separation.

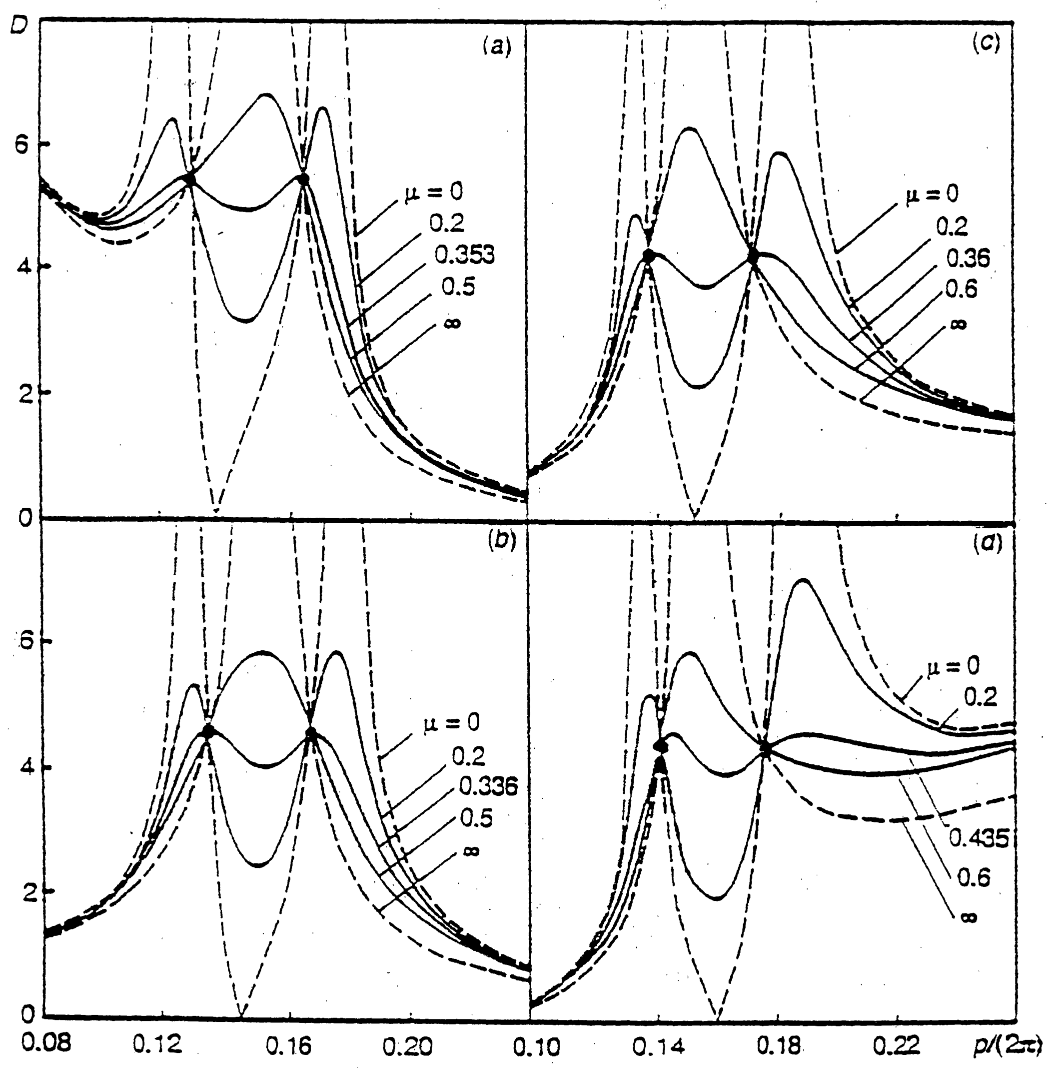

Figure 7 Amplitude and frequency diagram for various values of the absorber's viscous friction coefficient μ, with ν=0.1 and for (a) α=−2, f^2=0.735; (b) α=0, f^2=0.827; (c) α=2, f^2=0.909; (d) α=4, f^2=1.

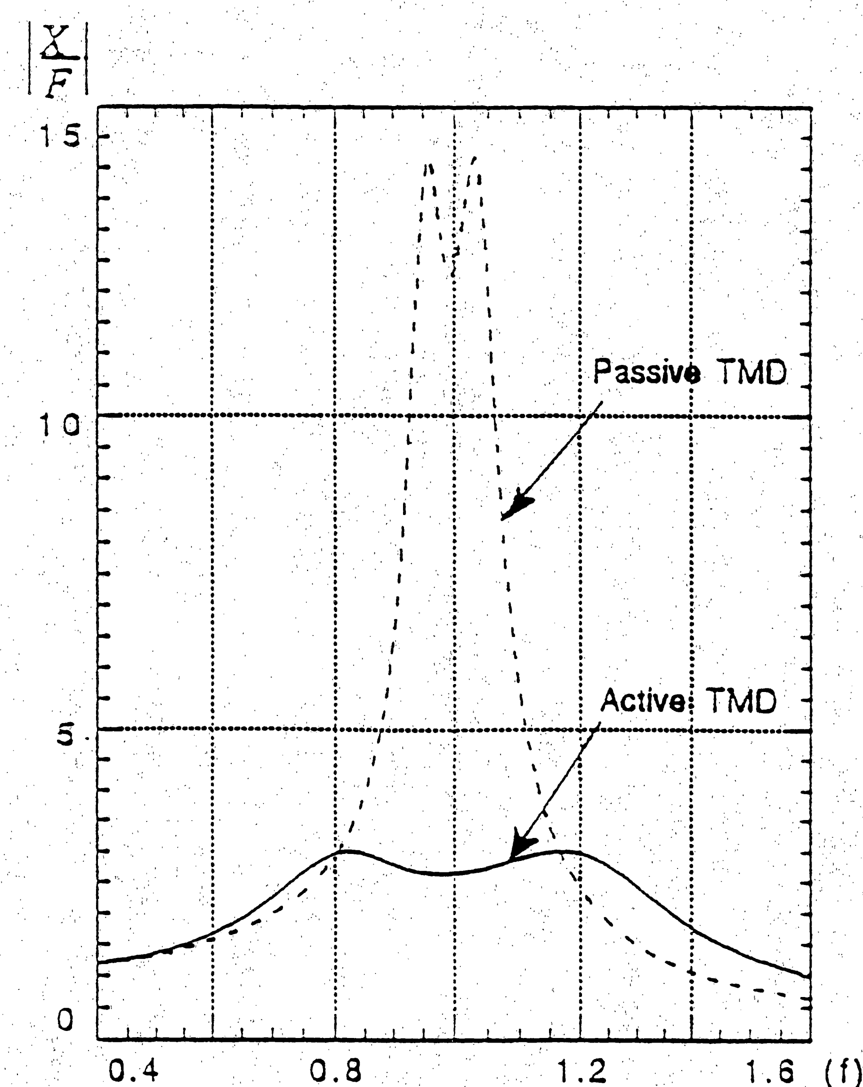

Figure 8 the comparison between tuned mass damper and proofmass damper when the mass ratio is 0.01.

Material	Viscous Damping Ratio
Aluminum	0.00005
Brass, bronze	<0.0005
Brick	0.005 - 0.01
Concrete	
Light	0.0075
Porous	0.0075
Dense	0.005 - 0.025
Copper	0.001
Cork	0.065 - 0.085
Glass	0.0003 - 0.001
Gypsum board	0.003 - 0.015
Lead	0.00025 - 0.001
Magnesium	0.00005
Masonry blocks	0.0025 - 0.0035
Oak, fir	0.004 - 0.005
Plaster	0.0025
Plexiglass, Lucite	0.01 - 0.02
Plywood	0.005 - 0.0065
Sand, dry	0.3 - 0.6
Steel, iron	0.00005 - 0.0003
Tin	0.001
Wood fiberboard	0.005 - 0.015
Zinc	0.00015

Table 1 Damping ratio for structural materials.

α	p_1^2	D_1^2	f_{opt}^2	μ_{opt}^2	a
-2	$\{3\sqrt{2+\nu}+\sqrt{2-7\nu} \mp \sqrt{[4+10\nu-2\sqrt{(2+\nu)(2-7\nu)}]}\} \times [4\sqrt{2+\nu}(1+\nu)]^{-1}$	$\dfrac{2(2+\nu)^2(1+\nu)^2}{\nu[2-\nu+\sqrt{(2+\nu)(2-7\nu)}]}$	$\dfrac{2-\nu+\sqrt{(2+\nu)(2-7\nu)}}{4(1+\nu)^2}$	$\left[-2+6\nu-11\nu^2+5.75\nu^3 + \sqrt{\dfrac{2-7\nu}{2+\nu}}(2+10\nu-3\nu^2 -2.75\nu^3)\right] \times \{4(1+\nu)^3[\sqrt{(2+\nu)(2-7\nu)} -4\nu]\}^{-1}$	$-0.5+\frac{1}{16}[2-7\nu +3\sqrt{(2+\nu)(2-7\nu)}] \times \sqrt{\left[\dfrac{2+5\nu+\sqrt{(2+\nu)(2-7\nu)}}{(2+\nu)(1+\nu)\nu}\right]}$
-1	$\{3\sqrt{2+\nu}+\sqrt{2-3\nu} \mp \sqrt{[4+14\nu-2\sqrt{(2+\nu)(2-3\nu)}]}\} \times [4\sqrt{2+\nu}(1+\nu)]^{-1}$	$\dfrac{2(2+\nu)(1+\nu)}{\nu\left(1+\sqrt{\dfrac{2-3\nu}{2+\nu}}\right)}$	$\dfrac{2-\nu+\sqrt{(2+\nu)(2-3\nu)}}{4(1+\nu)^2}$	$\left[4+9\nu-30.5\nu^2+7.75\nu^3 + \sqrt{\dfrac{2-3\nu}{2+\nu}}(-4+11\nu-0.5\nu^2 -1.25\nu^3)\right] \times [4(1+\nu)^3(4-7\nu)]^{-1}$	$-0.25+\frac{1}{16}[3(2-3\nu) +5\sqrt{(2+\nu)(2-3\nu)}] \times \sqrt{\left[\dfrac{2+7\nu+\sqrt{(2+\nu)(2-3\nu)}}{(2+\nu)(4+6.5\nu)\nu}\right]}$
0	$\dfrac{1\mp\sqrt{\nu/(2+\nu)}}{1+\nu}$	$\dfrac{2+\nu}{\nu}$	$\dfrac{1}{(1+\nu)^2}$	$\dfrac{3\nu}{2(1+\nu)^3}$	$\dfrac{1}{\sqrt{(2+\nu)\nu}}$
1	$\dfrac{[\sqrt{8+5\nu}\mp\sqrt{\nu}]^2}{4(2+\nu)(1+\nu)}$	$\dfrac{2+\nu}{\nu(1+\nu)}$	$\dfrac{1+0.5\nu}{(1+\nu)^2}$	$\dfrac{\nu(3+3\nu+0.625\nu^2)}{(2+\nu)(1+\nu)^3}$	$0.25+\sqrt{\dfrac{1+0.625\nu}{2\nu}}$
2	$\dfrac{1\mp\sqrt{\nu/[2(1+\nu)]}}{1+0.5\nu}$	$\dfrac{2}{\nu(1+\nu)}$	$\dfrac{1}{1+\nu}$	$\dfrac{3\nu}{(2+\nu)(1+\nu)}$	$0.5+\sqrt{\dfrac{1+\nu}{2\nu}}$
3	$\dfrac{[\sqrt{8+5\nu}\mp\sqrt{\nu}]^2}{2(2+\nu)^2}$	$\dfrac{2}{\nu(1+0.5\nu)^2}$	$\dfrac{1}{1+0.5\nu}$	$\dfrac{3\nu}{2-0.5\nu^2}$	$0.75+\sqrt{\dfrac{1+0.625\nu}{2\nu}}$
4	$1\mp\sqrt{\nu/(2+\nu)}$	$\dfrac{2}{\nu(1+0.5\nu)}$	1	$\dfrac{(3-\nu^2)\nu}{2(1-2\nu-\nu^2)}$	$1+\dfrac{1}{\sqrt{(2+\nu)\nu}}$

Table 2 Optimum results of a tuned mass damper.