Next Generation Very Large Array Memo No. 60

# Image Dynamic Range Limits Arising From Visibility Errors

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## 1 Introduction

Technical requirements for ngVLA design are being driven in part by the need to support high dynamic range imaging for deep field science.

Perley (1999) presents a comprehensive examination of issues relating to high dynamic range imaging, including the derivation of relationships between visibility errors and image dynamic range limits. However, the derivation does not consider the relationship between dual polarized receivers and the output Stokes image.

Given their importance to ngVLA design, in this memo I rederive the equations that relate visibility errors to image dynamic range limits and (in a minor contribution) complete the derivation for dual polarized receivers. For clarity and completeness, I reproduce most of the material from Section 3 of Perley (1999) and revise here accordingly.

The equations below are being used to drive technical requirements for aspects of ngVLA design that contribute to the visibility error budget, such as antenna pointing and tropospheric phase tracking capabilities.

## 2 The Effects of Visibility Errors on Image Dynamic Range

The following derivation uses simple arguments to allow rough calculation of dynamic range limits given baseline-based or antenna-based errors. For simplicity, the analysis will consider a point source at the phase center (for which all visibilities are the same) observed in a single frequency channel. The systematics of interest here (e.g. pointing, troposphere) will act coherently over any given observing band (which may comprise multiple channels within a bandwidth, or even a single channel with large fractional bandwidth), in which case the

*Revised Equations 7 and 10.

information content can be effectively modeled using only a single channel. Furthermore, the analysis will be performed in one dimension. The resulting equations will be valid for more complex sky brightness distributions observed with a two-dimensional array.

Consider a single integration observation (extreme snapshot) of a unit amplitude source located at the phase-tracking center, measured in a single polarization product (e.g. linear basis $XX$ or circular basis $RR$) in a single frequency channel. For an array with $N$ antennas there will be $N(N-1)/2$ complex visibilities. Suppose all but one are perfect. The unaffected visibilities have unit amplitude and zero phase and are described by $V(u) = \delta(u - u_k)$ for baseline lengths $u_k$ and where $\delta$ is the Dirac delta function. The discrepant visibility from baseline length $u_0$ is

$$V(u) = \delta(u - u_0)e^{-i\phi_b} \ , \tag{1}$$

where $\phi_b$ is the baseline-based phase error (in radians). The image is formed by evaluating the transform $I(l) = \int V(u)e^{i2\pi ul}du$, so for each 'good' baseline the integral gives a contribution of $2\cos(2\pi u_k l)$. The factor of two arises because each visibility is counted twice (once at $u_k$ and again at its complex conjugate). The 'bad' baseline contributes $2\cos(2\pi u_0 l - \phi_b)$, which for small $\phi_b$ becomes $2[\cos(2\pi u_0 l) + \phi_b \sin(2\pi u_0 l)]$. The resulting image is then

$$I(l) = 2\phi_b \sin(2\pi u_0 l) + 2 \sum_{k=1}^{N(N-1)/2} \cos(2\pi u_k l) \ , \tag{2}$$

while the beam (point spread function) is

$$B(l) = 2 \sum_{k=1}^{N(N-1)/2} \cos(2\pi u_k l) \ . \tag{3}$$

For a quasi-uniform distribution of spacings, the beam and image will both have amplitude $N(N-1)$ and width $\sim 1/u_m$ for maximum spacing in wavelengths $u_m$. Deconvolution is accomplished by subtracting the beam from the image, giving a residual $R(l) = 2\phi_b \sin(2\pi u_0 l)$, which is a periodic function of amplitude $2\phi_b$ and period $1/u_0$. If the dynamic range is defined as $D_{pq}$ = (peak on image)/(rms on image residual) for an image of a single polarization product (e.g. $pq = XX$), then

$$D_{pq} = \frac{N(N-1)}{\sqrt{2}\phi_b} \approx \frac{N^2}{\sqrt{2}\phi_b} \ , \tag{4}$$

with the approximation valid for large $N$.

The derivation for an amplitude error is similar. The visibility for the 'bad' baseline is written as $V(u) = (1 + \epsilon_b)\delta(u - u_0)$ for baseline-based amplitude error $\epsilon_b$. Following through, the same results are recovered with the substitution

$$\phi_b \to \epsilon_b \ . \tag{5}$$

The results above can now be generalized. Suppose that all baselines have an independent random error of the magnitude given above. Then, after modifying Equation 2, the dynamic range will be decreased from the single baseline case by a factor $\sqrt{N(N-1)/2}$, giving

$$D_{pq} = \frac{\sqrt{N(N-1)}}{\phi_b} \approx \frac{N}{\phi_b} \ . \tag{6}$$

Suppose instead that the errors are antenna-based and that these errors are uncorrelated between antennas. Following standard error propagation, the relationship between baseline-based and antenna-based phase errors is given by $\phi_b = \sqrt{2}\phi_a$. The relationship for amplitude errors is $\epsilon_b = \epsilon_a/\sqrt{2}$. If all antennas have typical phase errors of this magnitude, then the error contribution from Equation 2 must be a multiplied by $(N-1)/\sqrt{N-1}$, reflecting a randomized sine contribution over all baselines to a single antenna, and further multiplied by $\sqrt{N}$, reflecting the statistical addition of such errors from $N$ antennas. The dynamic range limit will therefore become

$$D_{pq} = \frac{\sqrt{N(N-1)}}{\sqrt{2}\phi_b} \approx \frac{N}{2\phi_a} \approx \frac{N}{\epsilon_a} \ . \tag{7}$$

For antenna-based errors dominated by effects that are uncorrelated between the orthogonal polarizations on an antenna (e.g. $X$ and $Y$), such as thermal noise, the polarization products (e.g. $XX$ vs $YY$) will be uncorrelated, in which case the dynamic range limit $D_s^{\text{pol-uncor}}$ for an image of a Stokes parameter (e.g. $s = I$, $Q$, $U$, or $V$) will improve to

$$D_s^{\text{pol-uncor}} \approx \sqrt{2} D_{pq} \ . \tag{8}$$

However, for antenna-based errors dominated by physical effects such as temperature changes in the antenna electronics, antenna pointing errors, or the atmosphere, the errors for the orthogonal polarizations on an antenna will be correlated. In this case, the dynamic range limit $D_s^{\text{pol-cor}}$ for an image of a Stokes parameter will remain

$$D_s^{\text{pol-cor}} \approx D_{pq} \ . \tag{9}$$

The effect of multiple integrations (multiple snapshots) can be estimated under the approximation that the errors vary over time. This can arise either from a change of the error, such as a changing atmosphere, or by rotation of the baseline, such that a particular $u$-$v$ coordinate will be measured by a different baseline with a different error. For $M$ successive independent snapshots, the dynamic range limit from Equation 7 becomes

$$D_{pq} \approx \frac{N\sqrt{M}}{2\phi_a} \approx \frac{N\sqrt{M}}{\epsilon_a} \ . \tag{10}$$

The value of $M$ depends on factors such as the atmospheric coherence timescale and the timescale for significant rotation of a baseline. Well motivated estimates for $M$ are important for estimating dynamic range limitations, though this is of less concern for large $M$ where differences are minimal due to the root dependence. Calculations to estimate $M$ are beyond the scope of this memo.

## References

Perley, R. A. 1999, Synthesis Imaging in Radio Astronomy II, 180, 275