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[Notes - NRAO Summer Lectures, 1969]

APERTURE SYNTHESIS

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Aperture synthesis is an elegant technique in which two antennas of variable separation up to some maximum baseline r_m are used to map the structure of radio sources with a resolution equivalent to that of a single dish antenna of diameter r_m. Since the engineering problems in building a fully steerable dish antenna of more than $\sim$400 ft diameter are formidable, aperture synthesis offers an alternative, extremely powerful and economical means of attaining very high angular resolution. The detailed mapping of radio sources with resolution approaching that of optical telescopes should bring a much better understanding of the complex physical processes occurring within them.

Let us begin by recalling some of the results derived at the end of the previous lecture ("Interferometry"). The response of an interferometer to an extended source with intensity distribution $I(x,y)$, where x and y are sky coordinates, was expressed by the function $R(u,v)$, where u and v are the coordinates of the interferometer baseline projected on the plane at right angles to the direction of the source. If the hour angle and declination of the centroid of the source are H and δ and the components of the baseline vector are B_x, B_y, B_z then u and v are given by equation (4). It was found that $R(u,v)$ may be expressed in terms of a complex function $\Gamma(u,v)$:

$$R(u,v) \propto \mathrm{Re}[\Gamma(u,v) e^{i\phi_o(u,v)}]$$

(equation (15)). Rewriting $\Gamma(u,v)$ as $\gamma(u,v) e^{i\phi(u,v)}$ we have

$$\therefore R(u,v) \propto \gamma(u,v) \cos(\phi(u,v) + \phi_o(u,v))$$

If we assume a position for the centroid of the source, $\phi_o(u,v)$ can be computed and so fitting the amplitude and phase of the cosinusoidal fringe output we can measure $\gamma(u,v)$ and $\phi(u,v)$.

From equations (11), (12), (14), and (15), we have

$$\Gamma(u,v) = \gamma(u,v)e^{i\phi(u,v)} = \frac{\iint I(x,y)e^{2\pi i(ux+vy)}dx\ dy}{\iint I(x,y)\ dx\ dy} .$$

This expression is known as the "Fourier Transform" of the intensity (or "brightness") distribution of the source and the function $\Gamma(u,v)$ is called the "Complex Visibility Function". We have now derived one of the two fundamental results upon which the principle of Aperture Synthesis depends:

I The output of an interferometer measures the Fourier transform of the brightness distribution of a source.

Let us now make a short digression into Fourier theory to get a feel for this relationship between source brightness distribution and its visibility function, and to derive a few important theorems that are relevant to aperture synthesis. There are many texts on Fourier Transforms: two suitable ones (both written by radio astronomers!) are:

Bracewell - "The Fourier Transform and Its Applications"

Jennison - "Fourier Transforms and Convolutions for the Experimentalist"

Here we shall for simplicity consider only one-dimensional transforms. The results may readily be extended to the two-dimensional case.

<u>Basic Fourier Transform Theory</u>

The Fourier Transform $\Gamma(u)$ of the function $I(x)$ is defined by

$$\Gamma(u) = \int_{-\infty}^{+\infty} I(x)e^{2\pi i\ ux}\ dx$$

It can be shown that the transformation can be "inverted", i.e.,

$$\int_{-\infty}^{+\infty} \Gamma(u)e^{-2\pi iux}du = \int_{-\infty}^{+\infty}\left[\int_{-\infty}^{+\infty} I(x)e^{2\pi iux}\ dx\right]e^{-2\pi iux}\ du$$

$$= I(x) .$$

This result is known as "Fourier's integral theorem". The proof is given in texts on Fourier theory. This is the second fundamental result on which Aperture Synthesis depends:

II **The source brightness distribution may be recovered by inverse Fourier transformation of the complex visibility function.**

Several symmetry relations between a function and its transform can be demonstrated. In our case we have the condition that $I(x)$, the source brightness distribution, is a real function of x. Let us see what symmetry condition this implies on $\Gamma(u)$.

Writing $\Gamma(u) = \Gamma_r(u) + i\Gamma_i(u)$, we have

$$I(x) = \int_{-\infty}^{+\infty} (\Gamma_r(u) + i\Gamma_i(u))\ (\cos 2\pi ux - i \sin 2\pi ux)\ du$$

$$= \int_{-\infty}^{+\infty} (\Gamma_r \cos 2\pi ux + \Gamma_i \sin 2\pi ux)\ du$$

$$- i \int_{-\infty}^{+\infty} (\Gamma_r \sin 2\pi ux - \Gamma_i \cos 2\pi ux)\ du.$$

We require that the imaginary term be zero for all x. By change of variable we have, for this term:

$$\int_0^{\infty} (\Gamma_r(u) - \Gamma_r(-u)) \sin 2\pi ux\ du$$

$$- \int_0^{\infty} (\Gamma_i(u) + \Gamma_i(-u)) \cos 2\pi ux\ du.$$

The condition that each term separately be zero for all x is

$$\Gamma_r(u) = \Gamma_r(-u)$$
$$\Gamma_i(u) = -\Gamma_i(-u)$$

or

$$\boxed{\Gamma(-u) = \Gamma^*(u)}$$

where the asterisk denotes the complex conjugate. This type of function is said to be "Hermitian". The importance of this result in Aperture Synthesis is that the complex visibility function Γ need only be determined over one half-plane. The function in the other half-plane can be constructed by means of the symmetry relation above, i.e., reflect through the origin and take the complex conjugate.

We now consider the transforms of some simple functions which we may regard as models of a source brightness distribution. Figure 1 is a pictorial display of the model $I(x)$ and its transform $\Gamma(u)$.

<u>(a) Point source at origin.</u>

We can represent a point source by the delta function $\delta(x)$ which has the properties:

$$\int_{-\infty}^{+\infty} \delta(x)\, dx = 1 \quad \text{and} \quad \int_{-\infty}^{+\infty} \delta(x-x_o)\, f(x)\, dx = f(x_o).$$

Setting $I(x) = \delta(x)$ gives

$$\Gamma(u) = \int_{\infty}^{\infty} \delta(x)\, e^{2\pi iux}\, dx = 1 \qquad \text{(Fig. 1(a))}$$

<u>(b) Point source at x_o</u>

Putting $I(x) = \delta(x-x_o)$ gives

$$\Gamma(u) = \int_{-\infty}^{\infty} \delta(x-x_o)\, e^{2\pi iux}\, dx = e^{2\pi iux_o}$$

$$= \cos(2\pi ux_o) + i \sin(2\pi ux_o) \qquad \text{(Fig. 1(b))}$$

This result is a simple example of the "Shift Theorem" which states that if $I(x)$ has a transform of $\Gamma(u)$, then $I(x-x_o)$ has a transform $e^{2\pi iux_o}\,\Gamma(u)$. That is, the amplitude of the visibility function is the same as for the unshifted source but there is now a "phase gradient" $\phi = 2\pi ux_o$ linearly increasing with u.

(c) Gaussian source at origin.

An extended source may conveniently be represented by the Gaussian function $I(x) = e^{-(x/a)^2}$.

$$\Gamma(u) = \int_{-\infty}^{+\infty} e^{-(x/a)^2 + 2\pi i u x}\, dx$$

$$= a\sqrt{\pi}\, e^{-\pi^2 a^2 u^2} . \qquad \text{(Fig.1(c))}$$

We have shown that the transform of a Gaussian is itself a Gaussian; if its half-width at the e^{-1} point of $I(x)$ is a, the half-width at e^{-1} of $\Gamma(u)$ is $1/\pi a$.

(d) Gaussian source at x_o

Now $I(x) = e^{-\left(\frac{x-x_o}{a}\right)^2}$, so

$$\Gamma(u) = \int_{-\infty}^{+\infty} e^{-\left(\frac{x-x_o}{a}\right)^2 + 2\pi i u x}\, du$$

$$= a\sqrt{\pi}\, e^{-\pi^2 a^2 u^2}\, e^{2\pi i u x_o} \qquad \text{(Fig.1(d))}$$

This result could of course have been derived immediately by application of the shift theorem.

(e) Double point source

A double point source may be represented by two delta functions, centered on $+ x_o$ and $- x_o$:

$$I(x) = \delta(x-x_o) + \delta(x+x_o)$$

$$\Gamma(x) = \int_{-\infty}^{\infty} [\delta(x-x_o) + \delta(x+x_o)]\, e^{2\pi i u x}\, dx$$

$$= e^{2\pi i u x_o} + e^{-2\pi i u x_o}$$

$$= \cos 2\pi u x_o \qquad \text{(Fig.1(e))}$$

(f) Double Gaussian source

For $I(x) = e^{-\left(\frac{x-x_o}{a}\right)^2} + e^{-\left(\frac{x+x_o}{a}\right)^2}$ we have

$$\Gamma(u) = a\sqrt{\pi}\, e^{-\pi^2 a^2 u^2} \cos 2\pi u x_o \qquad \text{(Fig. 1(f))}$$

by a combination of results (d) and (e).

We may consider this result as a special example of the "Convolution Theorem" which gives useful insight into the relationship between equivalent operations in the u-v and x-y planes.

The "Convolution" of function $f(x)$ by $g(x)$ is denoted by $f(x)*g(x)$ and defined by the integral

$$h(x) = f(x)*g(x) = \int_{-\infty}^{\infty} f(\xi)\, g(x-\xi)\, d\xi$$

A practical example from Radio Astronomy of how this integral can arise is afforded by the output from the receiver of a radio telescope with beam pattern $B(\xi)$ centered on point x on the sky and observing a source with brightness distribution $I(\xi)$. By inspection of Fig. 2 the output can be written as

$$R(x) \propto \int_{-\infty}^{\infty} I(\xi)\, B(x-\xi)\, d\xi.$$

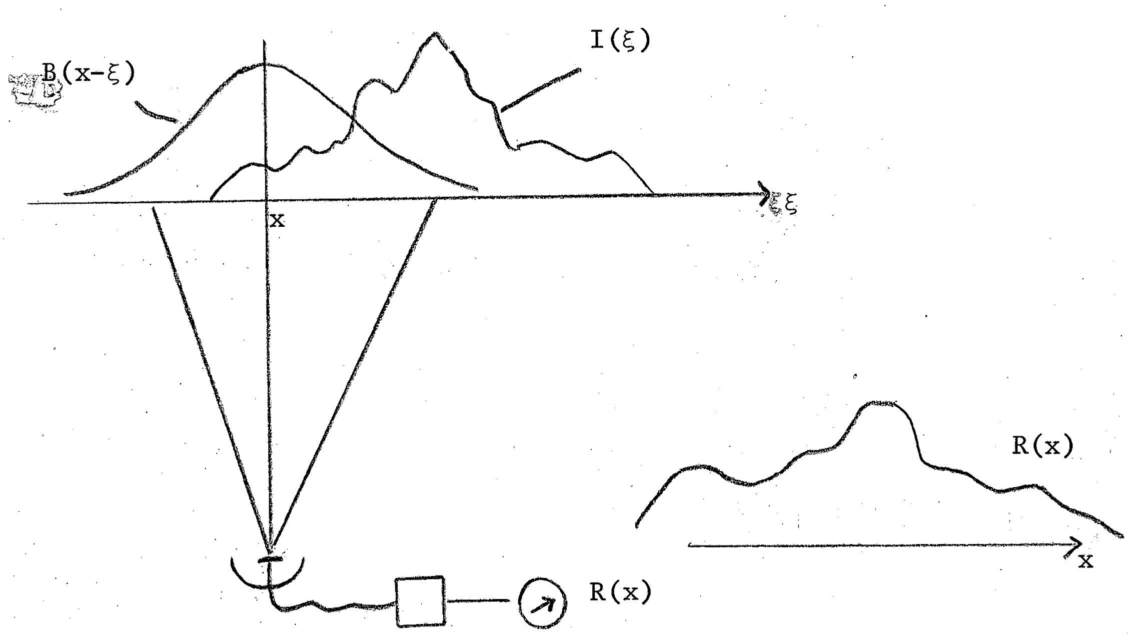


Fig. 2

That is, the output of the receiver is proportional to the convolution of the source brightness distribution with the telescope beam pattern.

Denote the Fourier transforms of f(x), g(x), and h(x) by F(u), G(u), and H(u). Then

$$H(u) = \int_{-\infty}^{\infty} \left[\int_{\infty}^{\infty} f(\xi)\, g(x-\xi)\, d\xi \right] e^{2\pi iux}\, dx$$

$$= \int_{-\infty}^{\infty} f(\xi) \left[\int_{\infty}^{\infty} g(x-\xi)\, e^{2\pi iux}\, dx \right] d\xi$$

$$= \int_{-\infty}^{\infty} f(\xi)\, e^{2\pi iu\xi}\, G(u)\, d\xi$$

$$= F(u)G(u).$$

This is the "Convolution Theorem." Denoting the transform operation now by a bar, we have

$$\overline{f(x)*g(x)} = F(u)G(u)$$

In words: "The Fourier transform of the convolution of two functions is equal to the product of their transforms."

To continue with the example above, we can use this theorem to see the relation between Aperture Synthesis and radio astronomy with single-dish telescopes.

In practice, the interferometer will sample the visibility function $\Gamma(u)$ in some discrete way. Let us define a "sampling function" S(u) so that the observations we have are of $\Gamma(u)S(u)$. Taking the transform gives us a map of the source

$$R'(x) = \overline{\Gamma(u)S(u)} = \overline{\Gamma(u)}*\overline{S(u)}$$

$$= I(x)*\overline{S(u)}$$

In the single-dish case, we had

$$R(x) = I(x)*B(x)$$

so that our "synthesized map" $R'(x)$ is the same as the map $R(x)$ we would get with a single-dish telescope having a beam pattern of $\overline{S(u)}$, the transform of our sampling function.

This brings us to the topic of sampling and the important "Sampling Theorem". Again, considering only the one-dimensional case, it is convenient to define a sampling function denoted by $III(u)$ which is a row of delta functions of separation Δu:

$$III(u) = \sum_{n=-\infty}^{\infty} \delta(u-n\Delta u) \qquad \text{(Fig. 3(a))}$$

The Fourier transform of $III(u)$ is given by

$$\overline{III(u)} = \int_{-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \delta(u-n\Delta u)\, e^{2\pi iux}\, du$$

$$= \sum_{n=-\infty}^{\infty} e^{2\pi in\Delta ux}$$

$$= \sum_{n=-\infty}^{\infty} \cos 2\pi n\Delta ux$$

$$+ i \sum_{n=-\infty}^{\infty} \sin 2\pi n\Delta ux$$

The second term vanishes by symmetry. The first term can be shown by the theory of Fourier series to be a row of delta functions with separation Δx given by $\Delta x = 1/\Delta u$. That is, the Fourier transform of $III(u)$ is itself a III function $III(x)$ (Fig. 3(b)).

Now suppose we have a source of brightness distribution I(x), falling to zero for $x > x_m$, and with transform $\Gamma(u)$ (Figs. 3(c) and (d)). Sampling of $\Gamma(u)$ at intervals of Δu is equivalent to taking the product of III(u) and $\Gamma(u)$ (Fig. 3(e)). By the convolution theorem, taking the transform of this product is the same as convolving I(x) with III(x) (Fig. 3(f)). We see that I(x) may be completely restored if $\frac{1}{\Delta u} > 2x_m$, or $\Delta u < \frac{1}{2x_m}$. This result is known as the "Sampling Theorem." We have shown that the sampling interval Δu in the u-plane determines the "field of view" x_m to which the source distribution may be mapped before repetition occurs and information is lost. In practice, the visibility function $\Gamma(u)$ can only be sampled to some maximum u_m set by the largest interferometer baseline available. By application of the sampling theorem, we see that the map derived need only be sampled at intervals $\Delta x = \frac{1}{2u_m}$. This is loosely equivalent to saying that we have independent samples at intervals on the sky of about one-half a beamwidth. Putting both results together, we have the number of samples of $\Gamma(u)$ observed

$$\frac{2u_m}{\Delta u} = 2\,\frac{\left(\frac{1}{2\Delta x}\right)}{\left(\frac{1}{2x_m}\right)} = \frac{2x_m}{\Delta x}$$

which is the number of independent samples of I(x) we can take.

The effect of sampling only to a maximum $u = u_m$ is to multiply III(u) by a "box function." (Fig. 4(a)). Its transform (exercise for the student!) is given by $2u_m\left(\frac{\sin 2\pi u_m x}{2\pi u_m x}\right)$ (Fig. 4(b)). We saw earlier that the "synthesized beam" is the transform of the sampling function, so that Fig. 4(b) represents the beam which repeats at intervals of $1/\Delta u$. The beam has negative sidelobes which are undesirable since they introduce physically unrealistic negative brightness into the map. This problem is reduced by "weighting" the observations with some function which reduces the amplitude of the higher spatial frequencies. A Gaussian weighting function which falls to zero at $u = u_m$, for example, would reduce the negative sidelobes to zero but now the beam is a Gaussian which is much broader than the unweighted beam. In practice, a weighting function is chosen which gives an acceptable compromise between increase of beamwidth and decrease of negative sidelobe level.

Observing Time and Sensitivity

(a) Observing time

It might be thought that much more observing time is required to map a given area of sky by the method of aperture synthesis than formapping with a single dish. That this is not so may quite easily be demonstrated.

Suppose we wish to synthesize a square antenna of side D using two small square elements A and B of side d. We have to make observations with A and B in all relative positions within the square area to completely sample the visibility function over that area. If we keep A fixed in one corner we would have to move B to all positions in the larger area shown in Fig. 5, about $2D^2/d^2$ positions in all. If we observe for a time τ (the integration time of the receiver used) at each position, the total time $T = \left(\frac{2D^2}{d^2}\right) \tau$.

However, we may now map (by Fourier inversion) an area of sky $\sim(\lambda/d)^2$. If we had a single square dish of side D we would have to observe points separated by $\frac{\lambda}{2D}$ on the sky, each for a time τ. The number of points within this area of sky is $(\lambda/d)^2/(\lambda/2D)^2 = (2D/d)^2$ so that the total time required is $T' = \frac{4D^2}{d^2} \tau$.

From this argument we see that the total observing time in the two cases is very similar.

(b) Sensitivity

Similarly, it might be thought that since much smaller antennae are used in the aperture synthesis case, the signal/noise ratio in the map would be much worse than in the single dish case. This assumption too is incorrect, as the following argument shows. The signal/noise ratio for an individual arrangement is proportional to the collecting area, $2d^2$. In synthesizing the map, $2D^2/d^2$ (=N) observations are combined in such a way that the signal at each point on the map is $\propto$ N and the noise is $\propto \sqrt{N}$ so that the final signal/noise ratio is proportional to $2d^2 \sqrt{N} = 2\sqrt{2}$ dD. In the single dish case, the signal/noise ratio is $\propto D^2$ so that the effective collecting area for synthesis is intermediate between that of the single dish and the individual elements of the interferometer.

In practice, it is much easier to obtain high sensitivity in a radio telescope than high resolving power, so in designing a synthesis telescope we may choose the size of the elements d so that the sensitivity achieved "matches" the resolving power. That is, in the synthesized map we are able to detect as many sources as we can separately resolve.

Brief Survey of Synthesis Telescopes and Results

(a) Early Cambridge systems

The synthesis technique has been exploited by the Cambridge group for many years. Some of the important early results are sky surveys, each made with a specially constructed instrument. For example, the 4C survey (Gower, Scott, and Wills, Mem. R.A.S., **71**, 49, 1967), the 38 MHz survey (Williams, Kenderdine and Baldwin, Mem. R.A.S., **70**, 33, 1966) and the Ryle-Neville North Pole survey (Ryle and Neville, Mon. Not. R.A.S., **125**, 39, 1962).

(b) Cambridge one-mile telescope

Comprises three 60-ft dishes on an E-W line. Two fixed and one movable on a half-mile railway track. With this telescope several "5C" surveys have been made (for example, 5C1: Kenderdine, Ryle and Pooley, Mon. Not. R.A.S., **134**, 189, 1966) and many strong sources have been mapped in considerable detail (e.g., Ryle and Neville, Nature, **205**, 1259, 1965; Macdonald, Kenderdine and Neville, Mon. Not. R.A.S., **138**, 259.

(c) CalTech interferometer

One-dimensional synthesis of many sources has been performed, both of unpolarized (Fomalont, Ap.J. Suppl. No. **138**) and polarized (Seielstad and Weiler, Ap.J. Suppl. No. **158**) radiation.

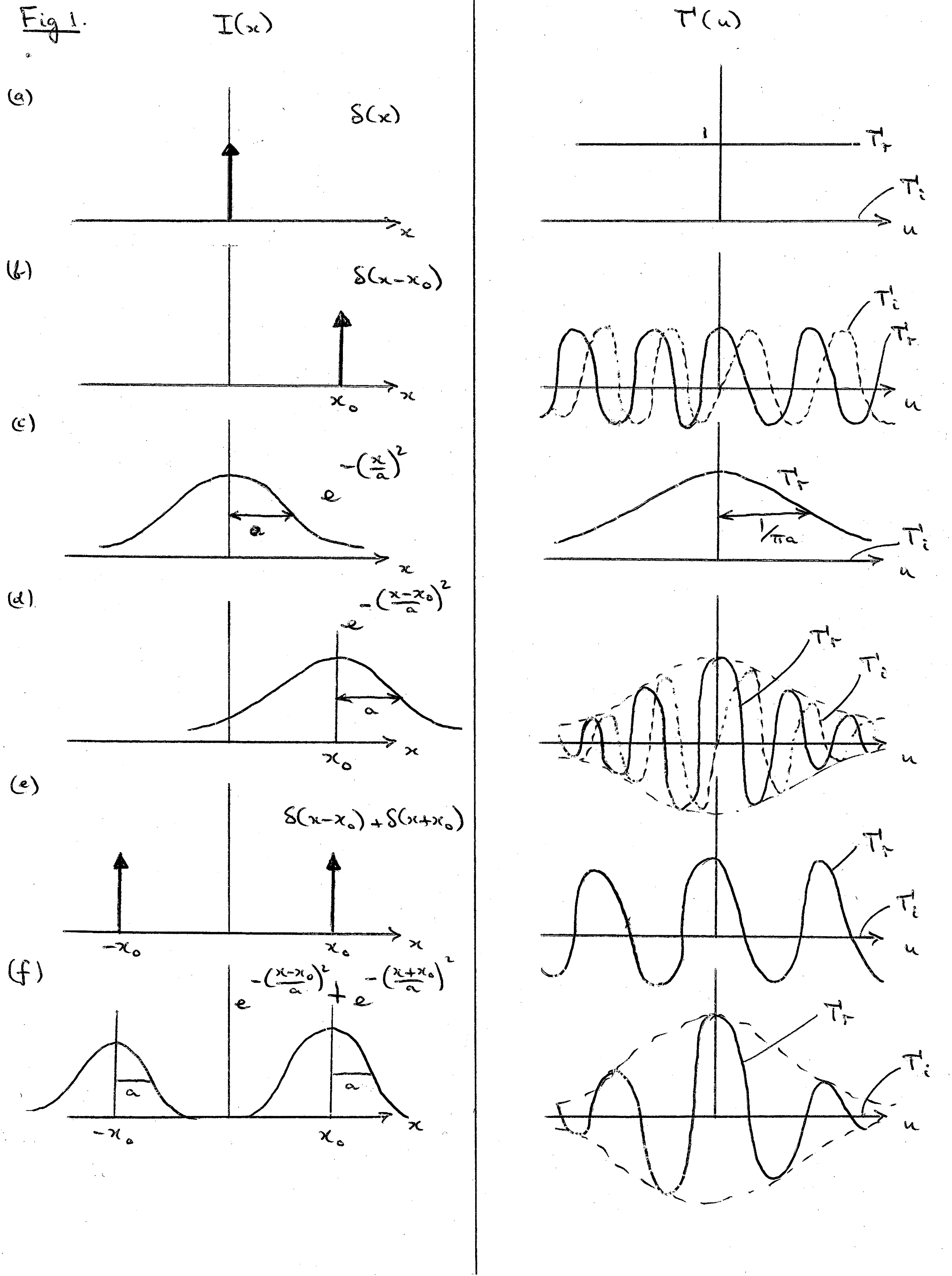
(d) NRAO interferometer

The NRAO interferometer has recently been used to synthesize maps of seven bright extragalactic sources and several H II regions.

Future synthesis telescopes

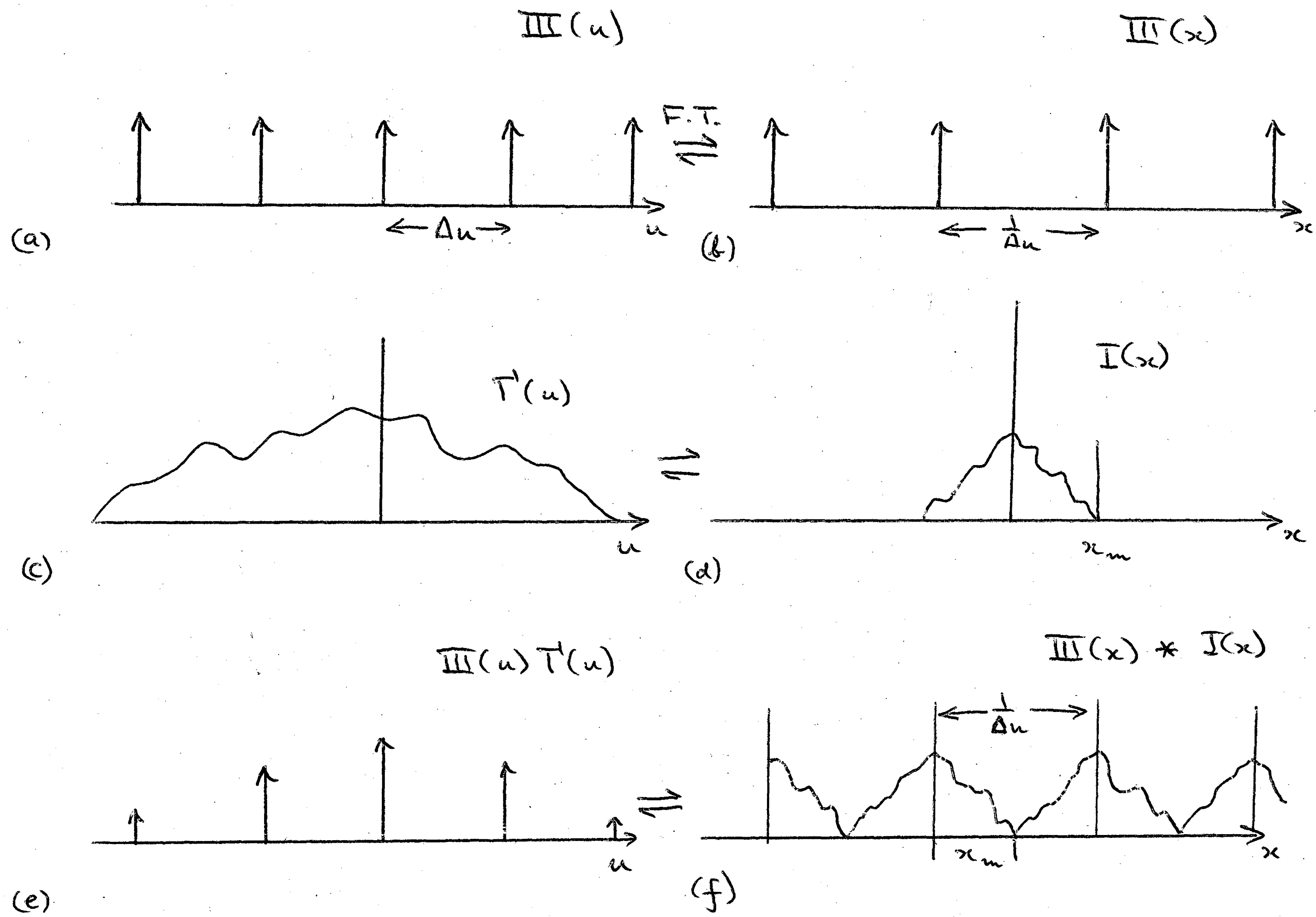
Large arrays to be used as aperture synthesis instruments are under construction in Holland (Westerbork), England (Cambridge), and Australia. There are arrays being planned by CalTech (8-element E-W) and NRAO (the VLA, 27-element Y-shaped configuration).

Fig 1.



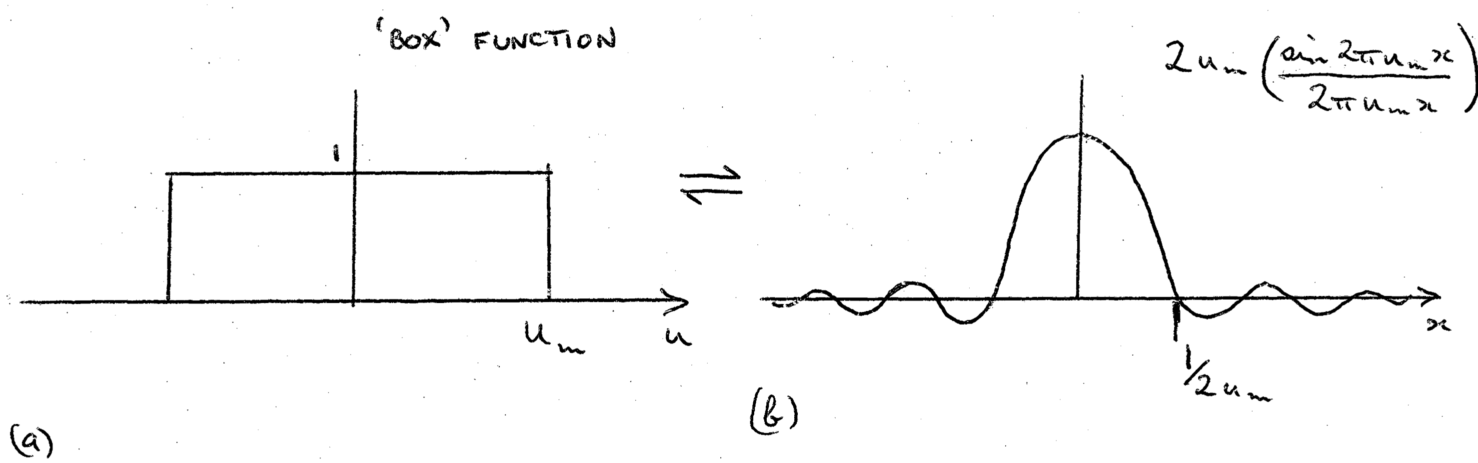
I(x)
T'(u)
(a)
δ(x)
x
1
T'_r
T'_i
u
(b)
δ(x−x_0)
x_0
(c)
e^{−(x/a)^2}
a
1/πa
(d)
e^{−((x−x_0)/a)^2}
(e)
δ(x−x_0) + δ(x+x_0)
−x_0
(f)
e^{−((x−x_0)/a)^2} + e^{−((x+x_0)/a)^2}

Fig. 3



III(u)
III(x)
F.T.
Δu
1/Δu
(a)
u
(b)
x
T'(u)
I(x)
(c)
(d)
x_m
III(u) T'(u)
III(x) * I(x)
(e)
(f)

Fig 4.



'BOX' FUNCTION
1
u_m
u
(a)
2u_m (sin 2πu_m x / 2πu_m x)
x
1/2u_m
(b)

Fig. 5.