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SYNCHROTRON RADIATION

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Interest in synchrotron radiation originated, obviously, from an investigation of radiation produced by synchrotron (and cyclotron) accelerators. Extraterrestrial applications did not occur until 1953 when Shklovsky proposed the synchrotron mechanism to explain the optical radiation from the Crab Nebula. Since then the process has been shown to be responsible for most of the non-thermal radiation from a majority of radiossources in the sky. Hence an understanding of synchrotron emission and the information it carries concerning the conditions of its origin is essential to work in radio astronomy.

We begin with a single non-relativistic charged particle and work our way up to an ensemble of relativistic particles imbedded in a magnetoactive plasma.

Non-relativistic Radiation

All accelerated charged particles radiate when moving in a vacuum. To see this we use the retarded potentials which allow for the finite propagation speed c of the E and B fields;

$$\phi(\vec{r},t) = \frac{e}{R-\vec{R}\cdot\vec{v}/c} = \frac{e}{s} \tag{1}$$

$$\vec{A}(\vec{r},t) = \frac{e\vec{v}/c}{R-\vec{R}\cdot\vec{v}/c} = \frac{e\vec{v}}{cs} \tag{2}$$

$$\frac{\partial}{\partial t} = \frac{R}{s}\frac{\partial}{\partial t'} \qquad \nabla = \nabla\big|_{ret\,t'} - \frac{Rc}{s}\frac{\partial}{\partial t'}$$

where $\vec{R} = \vec{r} - \vec{r}_c$, $\vec{r}$ being the position of the observer, $\vec{r}_e$ the position of the charged particle moving at $\vec{v}$.

Using

$$\vec{B} = \overline{\nabla} \times \vec{A} \tag{3}$$

$$E = -\overline{\nabla}\phi + \vec{c}\,\frac{\partial A}{\partial t} \tag{4}$$

gives

$$E = \frac{e}{s^3}\left(1 - \frac{v^2}{c^2}\right)\left(\vec{R} - \frac{R\vec{v}}{c}\right) + \frac{e}{c^2 s^3}\left\{ \vec{R} \times \left[\left(\vec{R} - \frac{R\vec{v}}{c}\right) \times \dot{\vec{v}}\right]\right\} \tag{5}$$

$$B = \frac{\overline{R} \times \overline{E}}{R} \tag{6}$$

The term not dependent on $\dot{\vec{v}}$ gives no radiation flux at infinity and thus no net radiation loss. The $\dot{\vec{v}}$ term does, and the radiaiton field drops as 1/R hence dominating the first term at large distances from the charge.

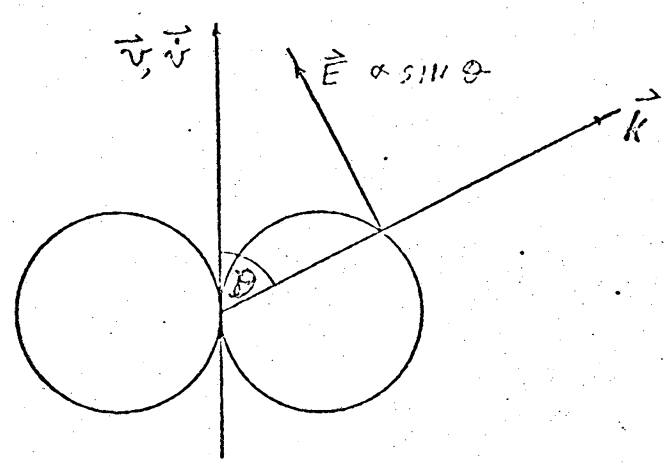
The energy flux is given by

$$\overline{S} = \frac{e}{4\pi}\,\overline{E} \times \overline{B} \tag{7}$$

and the angular distribution of radiated power is

$$\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3}\,\dot{v}^2 \sin^2\theta \tag{8}$$

where θ is the angle between $\dot{\vec{v}}$ and the direction to the observer. For $\dot{v} \parallel$ to $\vec{v}$ the distribution looks like:



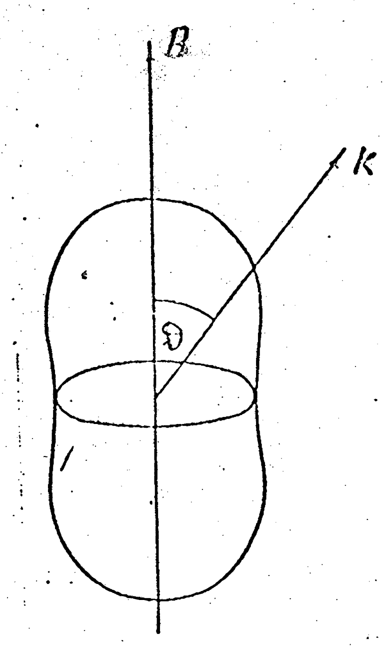
The total power is $P = 2\, e^2 \dot{v}^2/(3c^3)$. This is also the radiation pattern of an oscillating dipole. Larmor motion due to particles moving in a magnetic field is helical with

$$\omega_L = \frac{eB}{mc} \;, \quad R_L = \frac{vmc}{eB} \quad . \tag{9}$$

Thus we have acceleration, and the motion in a plane perpendicular to B (i.e., circular motion) can be regarded as a superposition of two dipoles oscillating $\pi/2$ out of phase. After averaging over the period we have

$$\frac{dP}{d\Omega} = \frac{e^2\, r_L^{\;2}\, \omega_L^{\;4}}{8\pi c^3} \; (1 + \cos^2 \theta) \tag{10}$$

where θ is the angle between $\vec{B}$ and the wave vector $\vec{k}$. The pattern appears as:

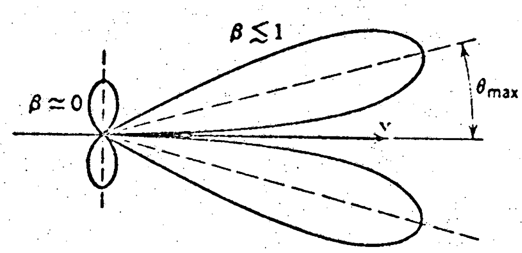


Relativistic Particles

To find the effects of relativistic motion, we can Lorentz transform the non-relativistic case into a frame moving at $v \sim c$. One finds in general that

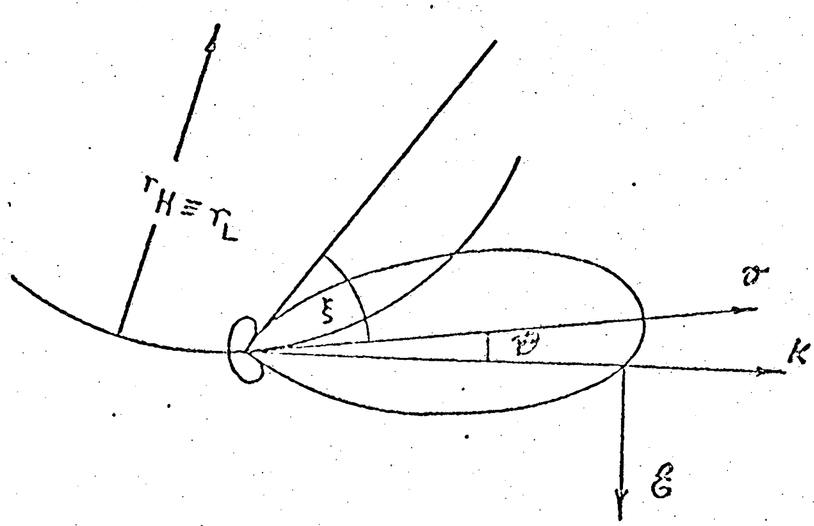
$$\frac{dP}{d\Omega} = \frac{e^2 \dot{v}^2}{4\pi c^3} \frac{\sin^2 \theta}{(1 - \beta \cos \theta)^5} \tag{11}$$

$\beta = v/c$ and the radiation pattern is



Radiation pattern for charge accelerated in its direction of motion. The two patterns are not to scale, the relativistic one (appropriate for $\gamma \sim 2$) having been reduced by a factor $\sim 10^2$ for the same acceleration.

with $\theta_{max} \sim \frac{1}{2\gamma}$ for $\gamma = \frac{1}{\sqrt{1-\beta^2}} >> 1$. For Larmor motion the radiation pattern becomes

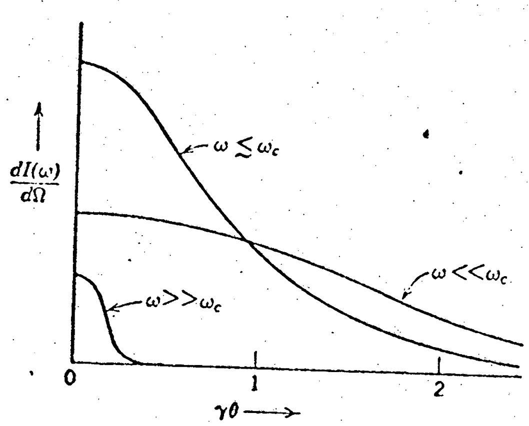


Projection of the electric field in a plane passing through the axis of a dipole as a function of the angle ϑ between the translational velocity v of the dipole and the wave vector k. The dipole is moving perpendicular to its axis, and the distribution of the field is shown for the case $v = \frac{2}{3}c$. The radius of curvature is r_H.

The radiation pattern and spectrum are obtained from eq. (5) by inserting motion appropriate to that of a charged particle in a magnetic field. Taking the Fourier transform of E and using Hermiticity one finds

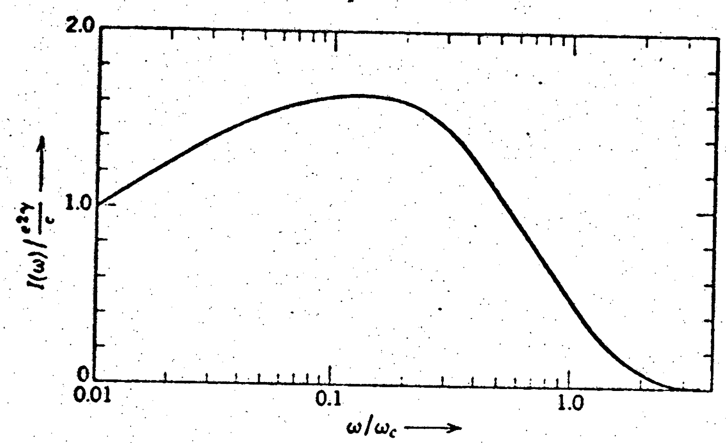
$$\frac{dI(\omega)}{d\Omega} = \frac{e^2}{3\pi^2 c}\left(\frac{\omega r_L}{c}\right)^2\left(\frac{1}{\gamma^2}+\theta^2\right)^2\left[K_{2/3}{}^2(x) + \frac{\theta^2}{1/\gamma^2+\theta^2}K_{1/3}^{2}(x)\right] \quad (12)$$

where $x = \frac{\omega r_L}{3c}\left(\frac{1}{\gamma^2}+\theta^2\right)^{3/2}$ and the K's are modified Bessel functions of the second kind. The $K^2_{2/3}$ term gives radiation polarized in the plane of the orbit, the $K^2_{1/3}$ term gives radiation polarized perpendicular to this. It turns out, after integrating over all angles, about seven times as much energy is radiated parallel to the plane as is perpendicular. What does all this look like? The beam is sharply peaked with a width $\sim 1/\gamma$. This width is a function of ω, however, as is seen below.



Differential frequency spectrum as a function of angle. For frequencies comparable to the critical frequency ω_c, the radiation is confined to angles of the order of γ^{-1}. For much smaller (larger) frequencies, the angular spread is larger (smaller).

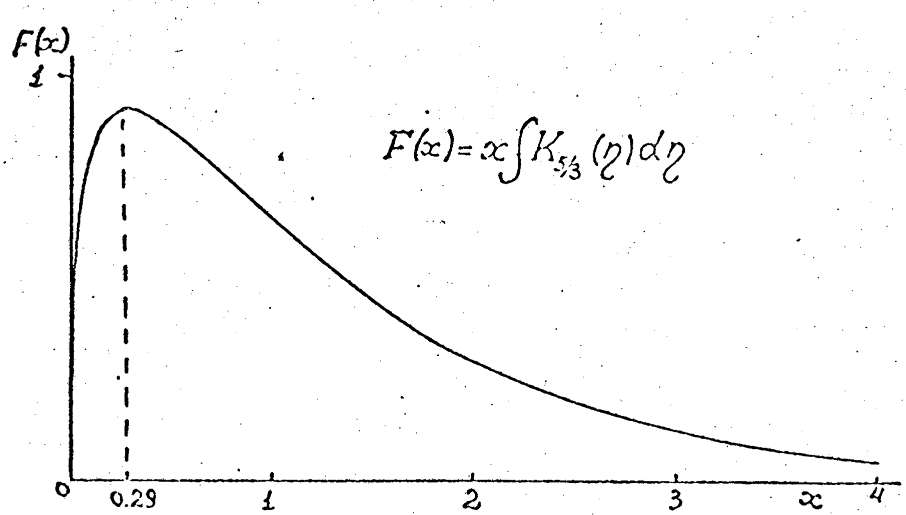
Here ω_c is defined as $\frac{3\ eB}{mc}\gamma^3$. The intensity integrated over all angles is shown below.



It is seen that the emission drops off rapidly above $\omega = \omega_c$.

It should be noted that another critical frequency $\nu_c \doteq \frac{3\ eB}{4\pi mc}\gamma^2$ has been defined by Ginzburg and Syravatskii and is often found in the literature.

The above figure on a more extensive, linear scale is shown below.



The spectral distribution of the power of the total (over all directions) radiation from charged particles moving in a magnetic field.

Here $x = \nu/\nu_c$.

Some short useful (but not always appropriate) formulae can be set down. The maximum frequency of radiation for particle of energy $E = \gamma mc^2$ is

$$\nu_m = .07 \frac{eB_\perp}{mc} \gamma^2 \text{ Hz} \tag{13}$$

$$= 1.8 \times 10^{18} B_\perp (E_{erg})^2 = 4.6 \times 10^{-6} B_\perp (E_{ev})^2$$

where the values in the last line are for electrons. $B_\perp$ is the value of B normal to the electron velocity.

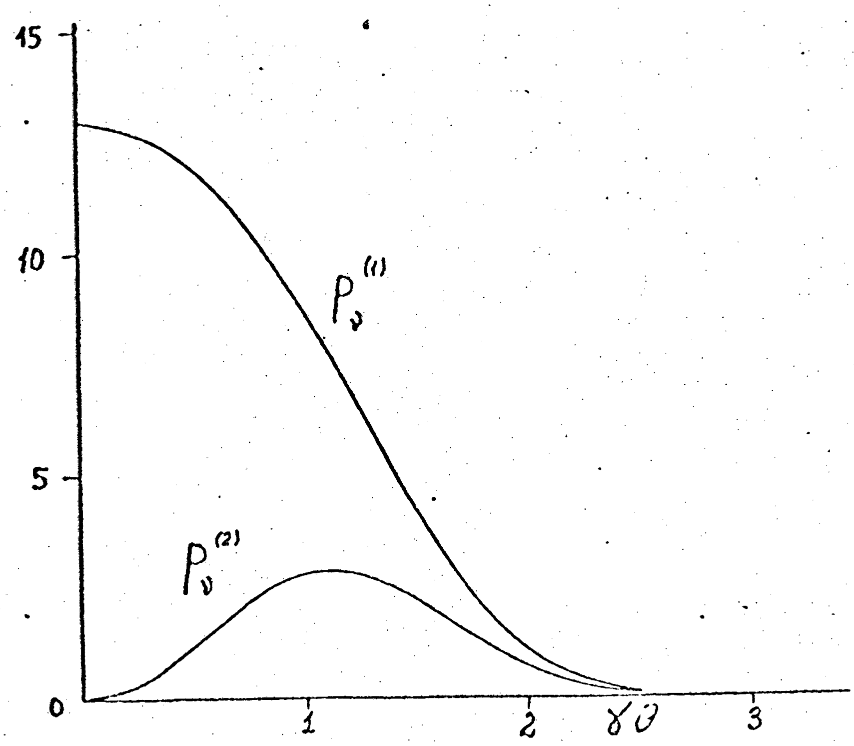
The total integrated power is

$$P(\gamma) = \frac{2 e^4 B_\perp^2}{3 m^2 c^3} (\gamma^2 - 1) \tag{14}$$

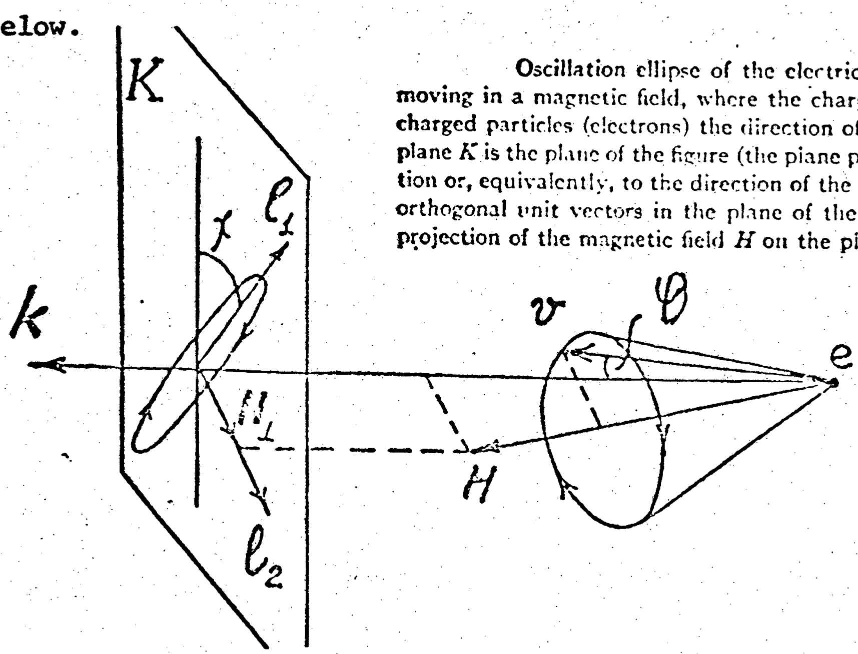
$$= 1.57 \times 10^{-15} B_\perp^2 \gamma^2 \text{ erg sec}^{-1}$$

again for electrons.

Finally, the polarization for the two perpendicular directions is given below.



It must be remembered that the helical motion of the particle produces a beam of radiation which describes a cone as the particle moves. The polarization vectors of the radiation in the observer's plane are given below.



Oscillation ellipse of the electric vector in a wave radiated by particles moving in a magnetic field, where the charge is taken as a positive. For negatively charged particles (electrons) the direction of rotation is opposite to that shown. The plane K is the plane of the figure (the plane perpendicular to the direction of the radiation or, equivalently, to the direction of the observer), and l_1 and l_2 are two mutually orthogonal unit vectors in the plane of the figure, of which l_1 is directed along the projection of the magnetic field H on the plane K.

Finally, the half life of a particle against synchrotron emission is

$$\tau = \frac{1}{E}\frac{dE}{dt} \approx \frac{5 \times 10^8 \, mc^2}{B_\perp^{\,2} \, E} \quad \text{sec} \qquad (15)$$

Ensembles of Particles

As radio sources consist of more than one charged particle (or two) we need to consider ensembles of charged particles with a given distribution in energy. Let the particle distribution be $N(E, \vec{r}, \vec{\Omega})\, dE\, d\Omega\, dV$, which is the number of particles in the volume dV with energy between E and E+dE and velocity inside the solid angle $d\Omega$ in the direction $\vec{\Omega}/\Omega$. We have found for each particle the intensity I as a function of ω, E, and θ. Let us assume for a moment the medium has no effect. Thus the total radiation intensity is a simple integral. The form for I used $B_\perp = B \sin\psi$ where ψ is the pitch angle. More generally we have the $I(E, \omega, \vec{r}, \theta, \psi)$. The radiation is emitted in a narrow cone directed along the instantaneous velocity of the particle, and the integration over $d\Omega$ is equivalent to integration over $d\theta$. Several other simplifying assumptions are now possible and usually made:

1. The relativistic particle distribution is homogeneous and isotropic, thus

$$N(E, \vec{r}, \vec{\Omega}) = \frac{1}{4\pi} N(E)$$

2. The magnetic field has random direction over a line of sight through the source and has constant amplitude. Thus one may average over ψ.

3. The distribution in energy for the particles is a power law:

$$N(E)\, dE = KE^{-\gamma}\, dE \ \mathrm{cm}^{-3}\ \mathrm{erg}^{-1}.$$

(n.b. γ hereafter is not $1/\sqrt{1-\beta^2}$.)

After these assumptions and considerable algebra one finds, for a source of dimension L along the line of sight, that

$$I_\nu = a(\gamma) \frac{e^3}{mc^2} \left(\frac{3e}{4\pi m^3 c^5}\right)^{(\gamma-1)/2} B^{(\gamma+1)/2}\ KL\nu^{-(\gamma-1)/2}$$

$$= 1.35 \times 10^{-22} a(\gamma) KLB^{(\gamma+1)/2} \left(\frac{6.26 \times 10^{18}}{\nu}\right)^{(\gamma-1)/2}$$

$$\mathrm{erg\ cm^{-2}\ sec^{-1}\ ster^{-1}\ Hz^{-1}} \qquad (16)$$

where the numerical values apply to electrons. This gives, under the above assumptions, the power crossing each cm^{-2} of the source surface per ster per Hz. The factor $a(\gamma)$ is

$$a(\gamma) = \frac{2^{(\gamma-1)/2}\sqrt{3}\ \Gamma\left(\frac{3\gamma-1}{12}\right) \Gamma\left(\frac{3\gamma+19}{12}\right) \Gamma\left(\frac{\gamma+5}{4}\right)}{8\sqrt{\pi}\ (\gamma+1)\ \Gamma\left(\frac{\gamma+7}{4}\right)} \qquad (17)$$

and ranges from $\approx$ .28 to .073 as γ goes from 1 to 4. Equation (16) shows that for a power law particle distribution $\propto E^{-\gamma}$, the intensity $I \propto \nu^{-\alpha}$, $\alpha = (\gamma-1)/2$. Such power law distributions in frequency are observed in many radio sources.

As observations are made on earth, the results are usually reported in flux density F_ν such as Watts m^{-2} Hz^{-1}. If R is the distance to the source and if R >> L,

$$F_\nu = \int I_\nu \, d\Omega = \frac{1.35 \times 10^{-22} a(\gamma) K \pi L^3 B^{(\gamma+1)/2}}{6R^2} \left(\frac{6.26 \times 10^{18}}{\nu}\right)^{(\gamma-1)/2} \quad W\,m^{-2}\,Hz^{-1} \qquad (18)$$

again for electrons.

A convenient form for the constant K (if all else is known) is obtained from the above equation, since F_ν is an observed quantity. We notice that all these forms give only the product of the particle energy and magnetic field to various powers. This is because the basic loss equation is $\propto E^2B^2$ per particle. Hence all the observations can give us is this product. The total energy in a radio source in both field and particles is minimized when the two energy densities are roughly equal. This "equipartition assumption" is usually used on the basis that nature will try to get away with something on a least cost basis.

If the magnetic field is not completely homogeneous and random, the above assumption (2) cannot be used. It turns out that for this case the net radiation is polarized due to field ordering. (Recall the polarization of an individual particle emission. If the field were completely ordered, this polarization would not be destroyed.)

The fractional polarization for a power law $N \propto E^{-\gamma}$ is

$$\pi = \frac{\gamma + 1}{\gamma + 7/3} \left(1 - \frac{2B_R^2}{3B_\perp^2} \right) \tag{19}$$

for the case of a strong random field B_R which is superimposed upon a homogeneous field B. $B_\perp$ is the projection of B upon the observing plane. For the case of a constant field strength B which has fluctuations $B_{\parallel}$ and $B_\perp$ relative to a symmetry axis,

$$\pi = \frac{15(\gamma+1)(\gamma+5)}{8(\gamma+7)(\gamma+7/2)} \frac{\overline{B_\parallel^2} - \overline{B_\perp^2}}{B^2} . \tag{20}$$

Influence of the Medium

The radiation emitted by an ensemble of relativistic particles must pass through those particles, as well as any other less energetic material that may be present, before it can escape. The presence of this material can affect the radiation through absorption processes. Radiation will not traverse a plasma below the plasma frequency

$$\omega_p = \left[\frac{4\pi ne^2}{m}\right]^{1/2} \tag{21}$$

so the presence of a lot of thermal gas can damp out the radiation.

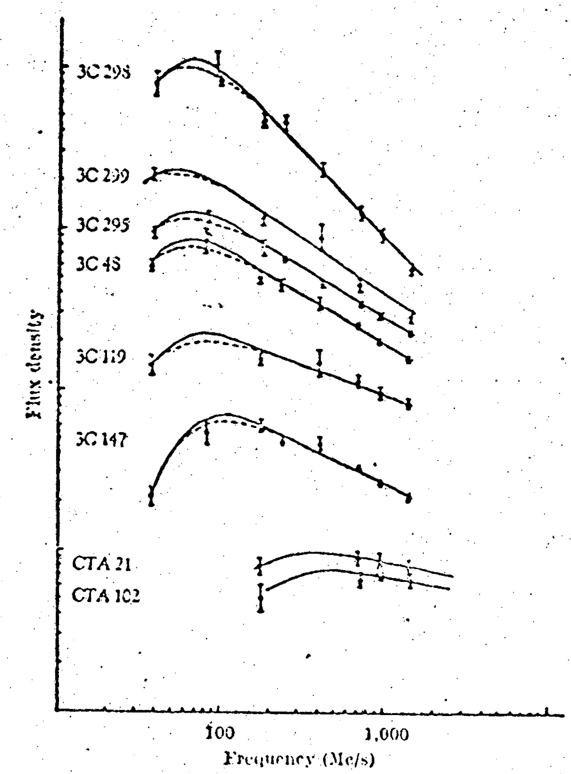
More importantly, the relativistic electrons themselves can reabsorb the radiation. The self absorption coefficient is given by

$$\mu = g(\gamma) \frac{e^3}{6\pi m} \left(\frac{3e}{2\pi m^3 c^5}\right)^{\gamma/2} KB^{(\gamma+2)/2} \nu^{-(\gamma+4)/2} \tag{22}$$

for a power law, where

$$g(\gamma) = \frac{3\sqrt{3}}{4} \Gamma\left(\frac{3\gamma+2}{12}\right) \Gamma\left(\frac{3\gamma+22}{12}\right) . \tag{23}$$

Thus for a layer of thickness L the intensity is $I_\nu \propto \frac{I_{\nu o}}{\mu}(1-e^{-\mu L})$ and if $L >> 1/\mu$, we have $I_\nu \propto \nu^{5/2}$ which is a very different spectrum than $\nu^{-\alpha}$. This effect can be very valuable in estimating the size of a radio emitting region from the shape of the spectrum as it goes from optically thick to optically thin. That is, for spectra which look like:



—, synchrotron absorption; - - -, thermal absorption

If ν_1 is the frequency at which turnover occurs and $F(\nu_1)$ is the flux density at that frequency, then the angular diameter is

$$\theta \approx 4 \times 10^{16} [F(\nu_1)]^{1/2} \nu_1^{-5/4} B_\perp^{1/4} (1+z)^{1/4}$$

where z is the redshift.