Interferometry: Lecture notes by Eric W. Greisen

I. Outline

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1. How a single dish works
2. Basic interferometry
3. How a real interferometer works

Day II
1. Mapping
2. An example of interferometry: Cyg A absorption
3. General question period

II References

A. Interferometry

a. Hjellming, R.M. 1973, An Introduction to the NRAO Interferometer - chapters 1 and 2 of this manual are of interest to non-users

b. Balick, B. 1973, Interferometry and Aperture Synthesis, Summer Student Lecture Notes 1973 - a good pictoral approach but contains a few serious errors

c. Fomalont, E.B. 1973 "Earth-Rotation Aperture Synthesis", Proc IEEE, 61, 1211 - available at Ivy Rd. + GB libraries only.

d. Read, R.B. 1963, Ap. J., 138, 1. - a basic reference on interferometer phase

B. Samples of the literature

a. Greisen, E.W. 1973, Ap. J., 184, 363 and 379 - line interferometry

b. Hogg, D.E. et al. 1969, A.J., 74, 1206 - continuum interferometry

c. Weiler, K.W., et al. 1971, Ap. J., 163, 455 - polarization interferometry

APPENDIX A

APERTURE SYNTHESIS

The method of interferometric aperture synthesis is best understood by first considering the analogy with the single-dish telescope. The voltage V produced at the horn of this telescope by an element of area of the dish dudv due to radiation at frequency ω from an element of sky dxdy may be written as

$$V(u,v,x,y,t)=V(x,y)W(u,v)e^{i[\omega t+\xi(x,y,t)]}\,e^{i2\pi[u(x-x_0)+v(y-y_0)]}$$

where the telescope is pointed at (x_0, y_0), u and v are measured in wavelengths, ξ is a random function which varies rapidly in all coordinates, and W is a weighting function which accounts for taper introduced by the horn and which is zero outside the dish area. The total voltage at the horn is the integral of this voltage over the dish and the sky. This voltage is then detected by a system which responds to the time average of the power. Thus

$$R(x_0, y_0) =$$

$$\beta^2 \int_{-\infty}^{\infty}\int dudv \int_{-\infty}^{\infty}\int du'dv' \int_{-\infty}^{\infty}\int dxdy \int_{-\infty}^{\infty}\int dx'dy' \; T^{\frac{1}{2}}(x,y)$$

$$T^{\frac{1}{2}}(x',y')W(u,v)W(u'v')\exp\left[2\pi i\{u(x-x_0)-u'(x'-x_0)+v(y-y_0)\right.$$

$$\left.-v(y'-y_0)\}\right] \langle \exp\left[i(\xi(x,y,t)-\xi(x',y',t)\right] \rangle$$

where we introduce the brightness temperature of the sky $T(x, y)$, where $T(x, y)$ is proportional to $V^2(x, y)$. Since

$$\langle e^{i\left[\xi(x,y,t)-\xi(x',y',t)\right]} \rangle = {}^2\delta(x-x',y-y')$$

by our assumption of a random noise signal, the response simplifies to

$$R(x_0, y_0) = \beta^2 \int_{-\infty}^{\infty}\int dudv \int_{-\infty}^{\infty}\int du'dv' e^{-2\pi i\left[(u-u')x_0+(v-v')y_0\right]}$$

$$W(u, v)\, W(u', v')$$

$$x \int_{-\infty}^{\infty}\int dxdy\; T(x,y)e^{i2\pi\left[(u-u')x+(v-v')y\right]} \quad . \tag{A-1}$$

Transforming coordinates we obtain finally

$$R(x_0,y_0) = \int\int_{-\infty}^{\infty} dudv\ F(u,v)A(u,v)e^{-2\pi i(ux_0+vy_0)} \tag{A-2}$$

where the complex fringe visibility function is defined as

$$F(u,v) \equiv \beta^2 \iint dxdy\ T(x,y)e^{i2\pi(ux+vy)}$$

and the autocorrelation of the weighting function is

$$A(u,v) \equiv \int\int_{-\infty}^{\infty} du'dv'\ W(u+u',\ v+v')\ W(u',v').$$

Thus, the single dish responds to the Fourier transform of the visibility function weighted by a sampling function.

Let us now consider the standard sampling theorem (Bracewell 1958). We sample the visibility function at regular intervals in u of $1/2\ x_c$ and v of $1/2\ y_c$. The sampled visibility function may be written as

$$V'(u,v) = {}^2\mathrm{III}(2x_c m\Delta u,\ 2y_c n\Delta v)\ V(u,v)$$

where the III (sha) function is defined as

$${}^2\mathrm{III}(x,y) = \sum_{\mu=-\infty}^{\infty} \sum_{\nu=-\infty}^{\infty} {}^2\delta(x-\mu,\ y-\nu).$$

If we Fourier transform the sampled visibility function we obtain a map given by

$$R(x, y) = {}^{2}\text{III}(x/2x_c, y/2y_c) * T(x, y)$$

where $T(x, y)$ is the true brightness distribution and $*$ denotes a convolution. In other words, we now have the desired map repeated at intervals of $2x_c$ in x and $2y_c$ in y. If

$$T(x, y) = 0 \text{ for } |x| > x_c \text{ or } |y| > y_c$$

then the repetition does not cause the maps to overlap and hence causes no error. Therefore, we must sample the visibility function at intervals smaller than the inverse diameter of the source being observed.

Since this sampling interval is about 400 feet (for the sources in this program at 21 cm) which is larger than the diameter of almost all single-dish telescopes, all information collected by the elements of a single dish is redundant for such sources. We may remove this redundancy or lack of resolution at reasonable cost if we sacrifice collecting area. We use one small telescope as the central surface element of the aperture to be synthesized while another movable telescope provides other surface elements of the synthesized aperture (Figure A1). If the voltages

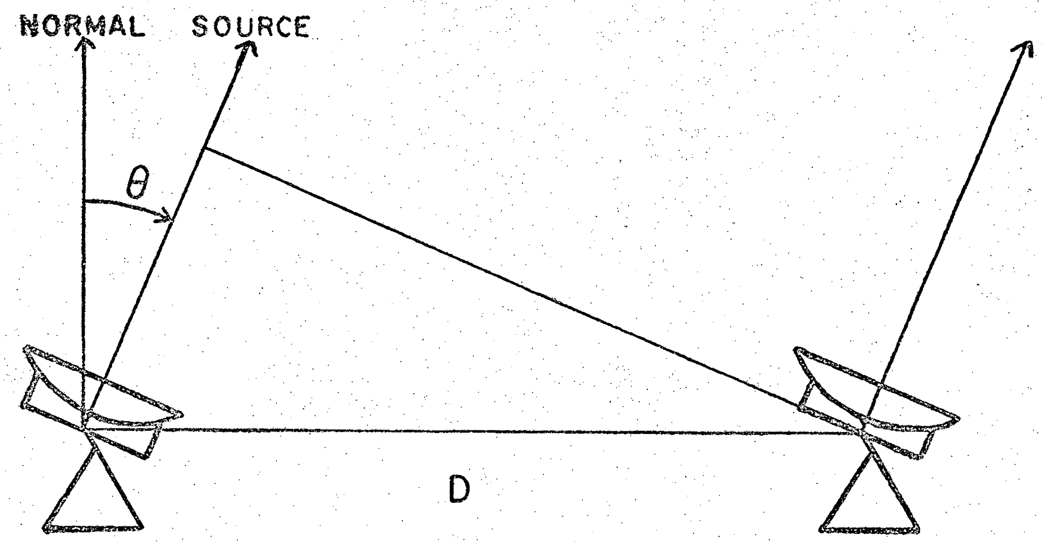


Figure A1. Geometry of two-element interferometer.

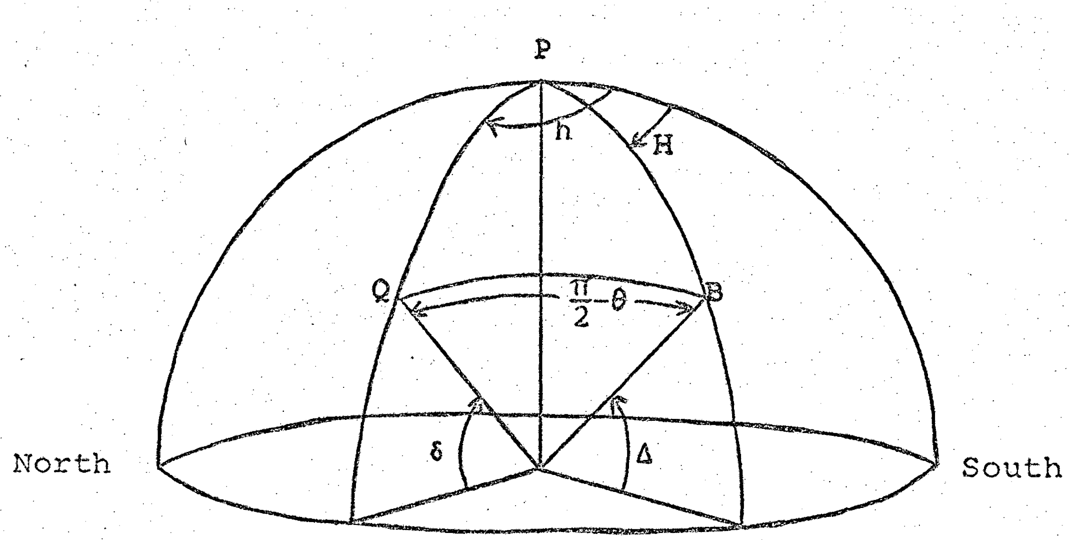


Figure A2. Celestial sphere with source (Q) and pole of baseline (B).

from the two telescopes are multiplied together and smoothed the output is

$$R(x,\ y) = V_0^{\ 2}\ e^{i\tau(x,\ y)} \qquad \text{(A-3)}$$

where $\tau(x,\ y) = \frac{2\pi D}{\lambda}\sin\theta(x,y)$ and V_0 is the voltage produced by the source at each telescope. As the earth rotates, $\sin\theta$ varies producing an approximately sinusoidal output called fringes. The phase τ may be expanded about some angle θ_0 as

$$\tau = \frac{2\pi D}{\lambda}\sin\theta_0 + \frac{2\pi D}{\lambda}\cos\theta_0\Delta\theta \qquad \text{(A-4)}$$

where the first term is the "expected fringe" which is removed during reduction and the second term is the "fringe phase". For an extended source, the total response of the interferometer is just the integral of the separate responses of equation (A-3) to produce

$$R(u,v) = \int_{-\infty}^{\infty}\!\!\int dxdy\ T(x,y)e^{i2\pi(ux+vy)}$$

where the expected fringe has been removed and the phase $\frac{D}{\lambda}\cos\theta_0\Delta\theta$ has been expressed in rectangular coordinates. In other words, the interferometer directly measures the visibility function.

The formulae for $\sin\theta_0$, u, and v may be derived as follows. The three points indicated on the celestial sphere in Figure A2 are the north celestial pole P, the pole of the extended baseline B and the source Q. The hour angle and declination of the source are h and δ, respectively, while the corresponding quantities for the baseline are H and Δ. The angle between Q and B is $\pi/2-\theta$. The law of cosines is used on the triangle PQB to give

$$\sin\theta = \sin\delta\sin\Delta + \cos\delta\cos\Delta\cos(h-H) .$$

The projection of the baseline along constant longitude is

$$v=-\frac{D}{\lambda}\frac{\partial\sin\theta}{\partial\delta}=-\frac{D}{\lambda}\left\{\cos\delta\sin\Delta-\sin\delta\cos\Delta\cos(h-H)\right\}$$

and the projection of the baseline along constant latitude is

$$u=\frac{D}{\lambda}\frac{1}{\cos\delta}\frac{\partial\sin\theta}{\partial h}=-\frac{D}{\lambda}\cos\delta\sin(h-H).$$

The choice of signs for u and v is a matter of convention while the factor $(1/\cos\delta)$ enters the equation for u so that x may be expressed in units of angle rather than time. As the source changes hour angle, the values of u and v describe an ellipse in the (u, v) plane. To fully sample the visibility function, we observe the source through the available range of hour angle at a number of discrete antenna separations D.

The Hermitian property of the visibility function is quite useful and easily derived. Since the visibility function and the source brightness are a Fourier-transform pair we may write

$$T(x,\ y)=\alpha \int_{-\infty}^{\infty}\!\!\int \mathrm{dudv}\ \underset{\sim}{V}(u,v)e^{-i2\pi(ux+vy)} \quad .$$

Writing the visibility function as the sum of a real and imaginary part we obtain the imaginary part of T as

$$\mathrm{Im}\big(T(x,\ y)\big) =$$

$$\alpha \int_{-\infty}^{\infty}\!\!\int \mathrm{dudv}\big[I(u,v)\cos 2\pi(ux+vy)-R(u,v)\sin 2\pi(ux+vy)\big] \ .$$

Since this must be zero we may use the symmetry properties of sines and cosines and the independence of R and I to show that

$$R(u,\ v) = R(-u,\ -v)$$

$$I(u,\ v) = -\ I(-u,\ -v)$$

or

$$V(u,\ v) = V^{*}(-u,\ -v) \ .$$

Therefore we need sample the visibility function over only one-half of the (u, v) plane. This formalism is only a more elegant way to express the fact that the only

How a real interferometer works: development of the basic equation

Refer to Figure A-1

Let:

x, y_1, y_2, z	be the electrical lengths of telescope 1 cables
$x+\Delta x, y_1+\Delta y_1, y_2+\Delta y_2, z+\Delta z$	be the corresponding electrical lengths for telescope 2 cables
ω_{LO}	be angular frequency 1st local oscillator
ω_2	be the sum of angular frequencies of later oscillators including synthesizer
ω_S	be the sum of angular frequencies which vary from scan to scan (e.g. synthesizer)
ω_a	be angular frequency of 1st lobe rotator
ω_b	be angular frequency of 2nd lobe rotator
θ	be phase shift of 1st lobe rotator
α	be phase shift of 2nd lobe rotator
ω_{IF}	be angular frequencies to which amplifiers following the delay lines respond
τ	be delay of signal reaching telescope 2 from that reaching telescope 1
$A_U(\beta,\gamma)$ $(B_U(\beta,\gamma))$	be the voltage produced by source at (β,γ) in upper sideband of telescope 1 (2)
$A_L(\beta,\gamma)$ $(B_L(\beta,\gamma))$	be the voltage produced by source at (β,γ) in the lower sideband of telescope 1 (2)

Note: $A_{U/L}$ and $B_{U/L}$ are random signals so that the time average of $A_i(\beta,\gamma,\omega_{IF})\,B_j(\beta',\gamma',\omega_{IF}')$
$= G_i(\omega_{IF})\,T_i(\beta,\gamma)\,{}^3\delta(\omega_{IF}-\omega_{IF}', \beta-\beta', \gamma-\gamma')\,\delta_{ij}$
where G_i is the gain and T_i the brightness temperature in the relevant sideband.

We now follow the voltages in the system. For telescope 1 they are the real parts of

$$\text{at (a)}\quad A = A_U \exp i\left[(\omega_{LO}+\omega_2+\omega_{IF})(t-x/c)\right] + A_L \exp i\left[(\omega_{LO}-\omega_2-\omega_{IF})(t-x/c)\right]$$

$$\text{at (b)}\quad L = \exp i\left[\omega_{LO}(t-y_1/c)\right]$$

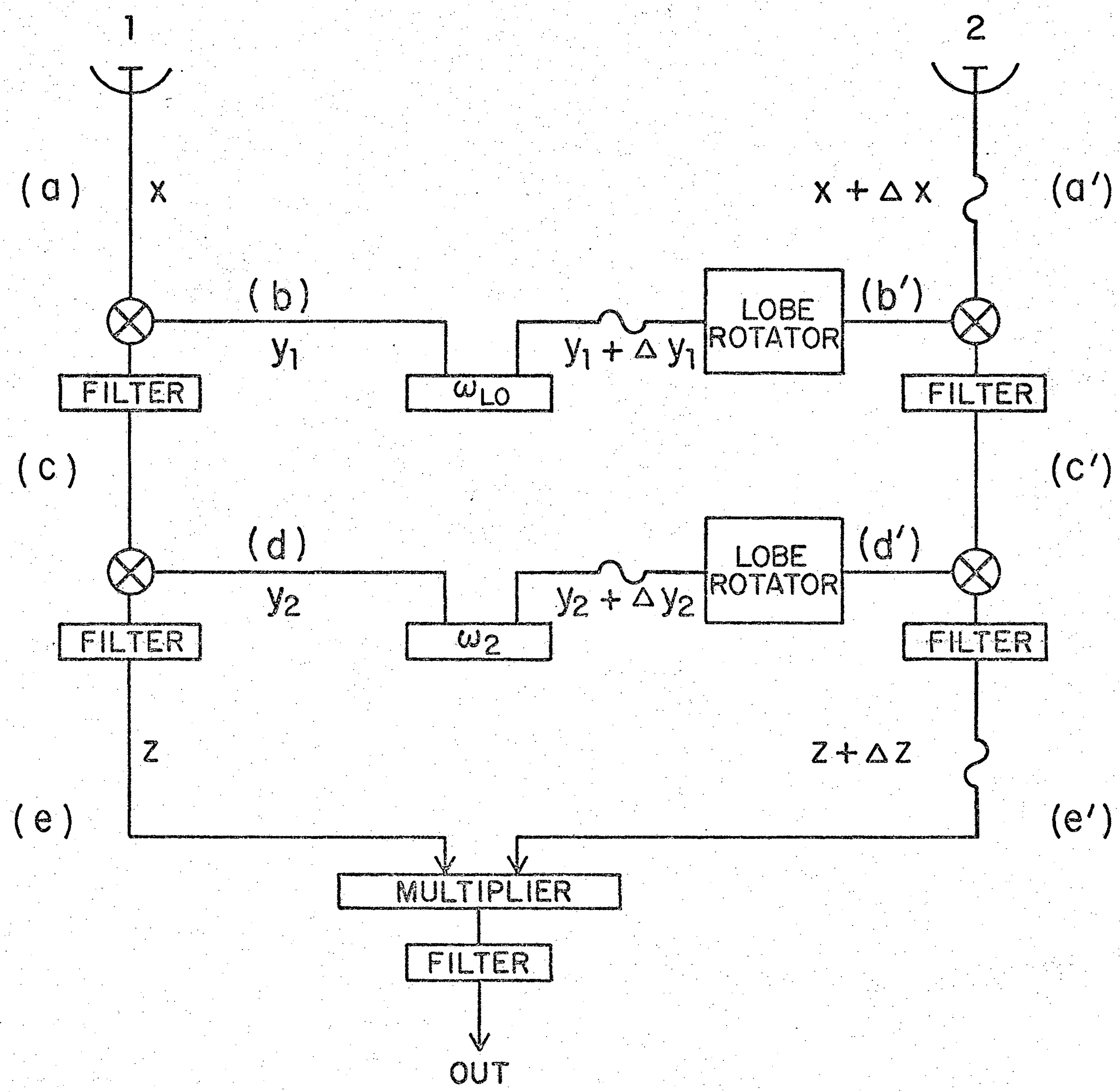


Figure A-1. Receiver Logic

The mixers multiply the signals at (a) and (b) with output $= AL^* + AL$. The AL term has frequency $2\omega_{LO} \pm \omega_2 \pm \omega_{IF}$ and is cut off by the filter leaving

at (c) $A = A_u \exp i\,[-\omega_{LO}(x/c - y_1/c) + (\omega_2 + \omega_{IF})(t - x/c)]$
$\quad + A_L \exp i\,[-\omega_{LO}(x/c - y_1/c) - (\omega_2 + \omega_{IF})(t - x/c)]$

(d) $L = \exp i\,[\omega_2 (t - y_2/c)]$

(e) $A = A_u \exp i\,[-\omega_{LO}(x/c - y_1/c) - \omega_2(x/c - y_2/c)$
$\quad + \omega_{IF}(t - x/c - z/c)]$
$\quad + A_L \exp i\,[-\omega_{LO}(x/c - y_1/c) + \omega_2(x/c - y_2/c)$
$\quad - \omega_{IF}(t - x/c - z/c)]$

Note: these expressions should show integrals over ω_{IF} and over the space coordinates β and γ

For telescope 2 the signal is delayed by τ seconds relative to that reaching telescope 1 where $\tau = \tau(\beta - \beta_0, \gamma - \gamma_0)$ and where subscript 0 refers to assumed baseline and source parameters. Thus,

at (a') $B = B_u \exp i\,[(\omega_{LO} + \omega_2 + \omega_{IF})(t - \tau - x/c - \Delta x/c)]$
$\quad + B_L \exp i\,[(\omega_{LO} - \omega_2 - \omega_{IF})(t - \tau - x/c - \Delta x/c)]$

Lobe rotator 1 shifts phase by $-\omega_a \tau_0 + \theta$ where θ will be some multiple of $\pi/2$ yielding

at (b') $L = \exp i\,[\omega_{LO}(t - y_1/c - \Delta y_1/c) - \omega_a \tau_0 + \theta]$

at (c') $B = B_u \exp i\,[-\theta + \omega_a \tau_0 - \omega_{LO}\tau - \omega_{LO}(x + \Delta x - y_1 - \Delta y_1)/c$
$\quad + (\omega_2 + \omega_{IF})(t - \tau - x/c - \Delta x/c)]$
$\quad + B_L \exp i\,[-\theta + \omega_a \tau_0 - \omega_{LO}\tau - \omega_{LO}(x + \Delta x - y_1 - \Delta y_1)/c$
$\quad - (\omega_2 + \omega_{IF})(t - \tau - x/c - \Delta x/c)]$

Lobe rotator 2 shifts phase by $-\omega_b \tau_0 + \alpha$ where α will be some multiple of $\pi/2$ yielding at (d′) $L = \exp i\,[\omega_2(t - y_2 - \Delta y_2/c) + \alpha - \omega_b \tau_0]$

(e′) $$B = B_u \exp i\,[-\theta - \alpha - (\omega_{LO} + \omega_2)\tau + (\omega_a + \omega_b)\tau_0 - \omega_{LO}(x/c + \Delta x/c - y_1/c - \Delta y_1/c) - \omega_2(x/c + \Delta x/c - y_2/c - \Delta y_2/c) + \omega_{IF}(t - \tau - x/c - \Delta x/c - z/c - \Delta z/c)]$$
$$+ B_u \exp i\,[-\theta + \alpha - (\omega_{LO} - \omega_2)\tau + (\omega_a - \omega_b)\tau_0 - \omega_{LO}(x + \Delta x - y_1 - \Delta y_1)/c + \omega_2(x + \Delta x - y_2 - \Delta y_2)/c - \omega_{IF}(t - \tau - x/c - \Delta x/c - z/c - \Delta z/c)]$$

The output is the time smoothed product of AB^* or, taking advantage of the random nature of the signal,

$$R = \int_0^\infty d\omega_{IF} \iint_{\text{(BEAM)}} d\beta\, d\gamma\, \Big[G_u(\omega_{IF})\, T_u \exp i\,[\Phi_u + \theta + \alpha] + G_L(\omega_{IF})\, T_L \exp i\,[\Phi_L + \theta - \alpha] \Big] \qquad (1)$$

where

$$\Phi_{u,L} = (\omega_{LO} \pm \omega_2 \pm \omega_{IF})\tau - (\omega_a \pm \omega_b \pm \omega_{IF})\tau_0 + \omega_{LO}(\Delta x/c - \Delta y_1/c) \pm \omega_2(\Delta x/c - \Delta y_2/c) \pm \omega_{IF}(\tau_0 + \Delta x/c + \Delta z/c) \qquad (2)$$

From equations (1) and (2) we may derive almost all properties of real interferometers except those related to noise and other errors.

I. An interferometer measures the Fourier Transform of the brightness distribution:

From (2)

$$\Phi_{u,L} = (\omega_{LO} \pm \omega_2 \pm \omega_{IF})(\tau - \tau_0) + (\omega_{LO} - \omega_a \pm \omega_2 \mp \omega_b)\tau_0 + \omega_{LO}(\Delta x - \Delta y_1)/c \pm \omega_2(\Delta x - y_2)/c \pm \omega_{IF}(\tau_0 + \Delta x/c + \Delta z/c)$$

We may define β and γ, the sky coordinates, as centered on the assumed source position. In which case, for small β and γ,

$$\tau - \tau_0 = u\beta + v\gamma$$

where u and v are the projected baseline in seconds. Note - in normal notation, u and v are the projected baseline in wavelengths but I wish to keep the frequency dependences clear at this point.

$$R = \int d\omega_{IF}\, G_u(\omega_{IF}) \exp i\,\Big[(\omega_{LO} - \omega_a + \omega_2 - \omega_b)\tau_0 + \omega_{LO}(\Delta x - \Delta y_1)/c$$
$$+ \omega_2(\Delta x - \Delta y_2)/c + \omega_{IF}(\tau_0 + \Delta x/c + \Delta z/c) + \theta + \alpha\Big] \tag{3}$$
$$* \iint d\beta\, d\gamma\; T(\beta, \gamma, \omega_{LO} + \omega_2 + \omega_{IF}) \exp i\,\big[(\omega_{LO} + \omega_2 + \omega_{IF})(u\beta + v\gamma)\big]$$

+ similar term for lower side band

The integral $d\beta\, d\gamma$ is simply a Fourier transform. Thus for a narrow IF passband, the response R is proportional to the Fourier transform of the source brightness distribution. This transform is known as the fringe visibility function

$$V = V\big((\omega_{LO} + \omega_2 + \omega_{IF})u,\ (\omega_{LO} + \omega_2 + \omega_{IF})v,\ \omega_{LO} + \omega_2 + \omega_{IF}\big).$$

or, for short, $V = V(\omega_{obs}, u, v)$

II. The response of a finite passband system is modulated by the ~~IF passband~~: Fourier transform of the IF passband:

Rewrite (3) again as

$$R = \exp i\,\big[(\omega_{LO} - \omega_a + \omega_2 - \omega_b)\tau_0 + \omega_{LO}(\Delta x - \Delta y_1)/c + \omega_2(\Delta x - \Delta y_2)/c$$
$$+ \theta + \alpha\big]$$
$$* \int d\omega_{IF}\; G(\omega_{IF})\, V(\omega_{LO} + \omega_2 + \omega_{IF}, u, v) \exp i\,\big[\omega_{IF}(\tau_0 + \Delta x/c + \Delta z/c)\big]$$
$$+ \text{term for lower sideband}$$

Let F = Fourier transform of $G(\omega_{IF})\, V(\omega_{LO} + \omega_2 + \omega_{IF}, u, v)$ relative to ω_{IF}

Then

$$R = F_u(\tau_0 + \Delta x/c + \Delta z/c)\, \exp i\,[(\omega_{LO} - \omega_a + \omega_2 - \omega_b)\tau_0 + \omega_{LO}(\Delta x - \Delta y_1)/c + \omega_2(\Delta x - \Delta y_2)/c + \theta + \alpha]$$

$$+ F_L(-\tau_0 - \Delta x/c - \Delta z/c)\, \exp i\,[\omega_{LO} - \omega_a - \omega_2 + \omega_b)\tau_0 + \omega_{LO}(\Delta x - \Delta y_1)/c - \omega_2(\Delta x - \Delta y_2)/c + \theta - \alpha]$$

To maximize the response, observers normally insert cable, crystal, and/or digital delays in the system as part of the Δz term. The delays are varied with time under computer control s.t.

$$\Delta z = -\Delta x - c\tau_0$$

Then the response is evaluated at $F_{\frac{u}{L}}(0)$ rather than at $F_{\frac{u}{L}}$'s which vary with time. This operation is called tracking delay.

If the IF passband is narrow enough, i.e.

$$G(\omega_{IF}) = 0$$

except for a narrow ~~band~~ range of frequencies, then one may observe with fixed delays. In such a case, F is a very broad function and hence changes very little in <u>amplitude</u> from $F(0)$ to $F(\tau_0 + \Delta x/c - \text{delay})$.

In following sections I will discuss single, mixed and double sideband modes of operation. In the first two modes, fixed delay operation is possible and in some cases desirable. In these cases the bandwidths are often fairly narrow. However the phase is seriously - but correctably - affected by the use of fixed delay. This is why one might wish to keep the delay fixed. In ~~~~ tracking delay the delay moves in discrete steps of say one to three nanoseconds. Thus

$$\Delta z = -\Delta x - c\tau_0 + \varepsilon(t)$$

with ε some form of sawtooth :

The jumps in phase caused by jumps in ε may seriously affect the data on single sideband systems. No double sideband system really has a narrow bandpass since, in effect, the two sidebands will interfere with each other. Thus, delay is always tracked with double sideband systems.

III. Let us consider first a double sideband system in which we wish to observe ~~to~~ each sideband separately. I called this a mixed sideband system.
Equation (1) is

$$R = \text{Real part}\left[\iiint d\omega\, d\beta\, d\gamma \;\; G_u T_u e^{i(\Phi_u+\theta+\alpha)} + G_L T_L e^{i(\Phi_L+\theta-\alpha)}\right]$$

There are various possible useful combinations of θ and α:

θ	α	R
0	0	Real (u) + Real (L)
90	−90	Real (u) − Real (L)
90	0	− Imag (u) − Imag (L)
0	90	Imag (u) − Imag (L)

Thus with the two phase shifters operated in such a sequence we can obtain the cosine and sine parts of both sidebands separately. The mixed system is otherwise similar to a ~~do~~ single sideband system.

IV. Let us now consider a double sideband system such as the NRAO dual-frequency system:

For this system:

$$\omega_a = \omega_b = 0$$

$$\theta = \alpha = 0$$

$$G_u(\omega_{IF}) \approx G_L(\omega_{IF})$$

and, since, $\omega_{IF} << \omega_{LO}$

$$V(\omega_{LO}+\omega_{IF}) \approx V(\omega_{LO}-\omega_{IF}) \simeq V(\omega_{LO})$$

Then equation (1) becomes

$$R = V(\omega_{LO}) \exp i\left[(\omega_{LO}-\omega_a)\tau_0 + \omega_{LO}(\Delta x - \Delta y_1)/c\right]$$

$$* \int d\omega_{IF}\ G(\omega_{IF}) \left(e^{+i\omega_{IF}(\tau_0+\Delta x/c+\Delta z/c)} + e^{-i\omega_{IF}(\tau_0+\Delta x/c+\Delta z/c)}\right)$$

$$= V(\omega_{LO})\, e^{i\varphi} \int_0^\infty d\omega_{IF}\, 2G(\omega_{IF}) \cos\left[\omega_{IF}(\tau_0+\Delta x/c+\Delta z/c)\right]$$

where φ is an instrumental phase if $\omega_a = \omega_{LO}$. The modulation due to the interference of the two sidebands and within each sideband is in this special case a real quantity. Thus delay errors affect only amplitude in double sideband systems.

Note: since only the real part ~~is~~ actually received, it is necessary to do some phase shifting even in ordinary double sideband systems. The most common approach is to set

$$\omega_a = \omega_{LO} + \frac{\omega_0}{\tau_0}$$

where ω_0 is some constant such as $2\pi/60$ radians/sec. θ may also be used.

II. Let us now consider a single sideband system such as the VLA:

$$G_L(\omega_{IF}) = 0$$

$$\omega_2 = \omega_b = \alpha = 0$$

Equation ① becomes

$$R = \exp i\left[(\omega_{LO}-\omega_a)\tau_0 + \omega_{LO}(\Delta x - \Delta y_1)/c + \theta\right]$$

$$* \int d\omega_{IF}\ G(\omega_{IF}) \exp i\left[\omega_{IF}(\tau_0+\Delta x/c+\Delta z/c)\right]$$

$$* \iint d\beta\, d\gamma\ T_u(\omega_{IF}) \exp i\left[(\omega_{LO}+\omega_{IF})(u\beta + v\gamma)\right]$$

Usually for fixed delay $\omega_a = \omega_{LO} + \omega_{IF0}$ where ω_{IF0} is the center frequency of the IF passband. Then:

$$R = e^{[i(\Delta x - \Delta y_i)/c + i\theta]} \int d\omega_{IF}\, G(\omega_{IF})\, V(\omega_{LO}+\omega_{IF}, u, v)$$

$$\exp i\left[(\omega_{IF}-\omega_{IF0})(\tau_0 + \Delta x/c + \Delta z/c) + \omega_{IF0}(\Delta x/c + \Delta z/c)\right]$$

Since $(\omega_{IF} - \omega_{IF0}) << \omega_{IF0}$ usually, this choice of ω_a leads to a far less rapid variation of phase with time in fixed delay modes.

To simplify the equations, for the moment, assume fixed delay. If $\omega_a = \omega_{LO}$, we have

$$R = e^{i[(\Delta x - \Delta y_i)/c + \theta]} \iiint d\gamma\, d\beta\, d\omega_{IF}\; T(\omega_{LO}+\omega_{IF})\, G(\omega_{IF})$$

$$* \exp i\left[(\omega_{LO}+\omega_{IF})(u\beta + v\gamma)\right]$$

For long baselines (u, v large) and significant displacements from the phase center (β, γ large), the variation in phase across a wide IF passband of the phase term above can cause significant effects in the data. This problem constitutes a fundamental limitation for the VLA. With the full VLA array and full 50 MHz continuum bandwidth, this effect limits the field of view to a small fraction of the beamwidth of the individual antennas.

Part II

Mapping

We will return to more conventional notation in which u and v are the projected baseline components in wavelengths and x and y replace β and δ as the spatial components.

The interferometer measures a visibility function $V(u,v)$ which is the Fourier transform of the source brightness distribution. Since the universe is moderately well behaved $V(u,v)$ is a smooth function defined for all u, v. However, since astronomers are neither infinitely wealthy nor infinitely patient, they actually observe V only at some discrete set of points (u_j, v_j). In other words, they observe a sampled visibility function

$$V'(u,v) = V\,S$$

where $$S \equiv \sum_{j=1}^{M} {}^2\delta(u-u_j, v-v_j)\,\omega_j$$

with ω_j being the weight assigned the j'th observation. Some astronomers think they are wealthy and patient and hence use a direct (or brute force) Fourier transform on V' to obtain

$$T' = T * \overline{S}$$

where the upper bar denotes a Fourier transform, the $*$ a convolution and V and T are Fourier transform pairs. We'll return to the subject of the synthesized map ("dirty map") equalling the convolution of the real world with the synthesized beam pattern ("dirty beam").

Some astronomers are either poor or enamored with elegant computer algorithms. Such astronomers realize that the FFT is both more elegant and faster than the brute force Fourier transform. However, the FFT requires the data to lie on a rectangular grid while the (u_j, v_j) actually lie on elliptical arcs. Thus, these astronomers "smooth their data to a grid" to use their words. Mathematically this means they convolve (smooth) their data with some function C and then resample the smoothed V' with a rectangular sampling function, e.g.

$$V'' = \mathrm{III}(u,v) \cdot C(u,v) * \left(V(u,v) \cdot S(u,v)\right)$$

where $$\mathrm{III}(u,v) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} {}^2\delta(u - m\Delta u, v - n\Delta v)$$

The result after the FFT is

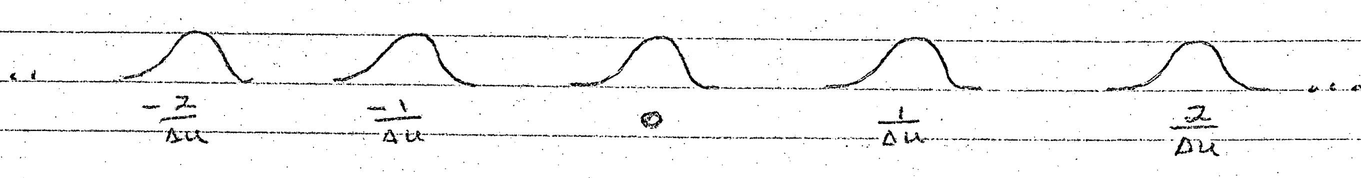
$$T'' = \overline{\mathrm{III}} * \left(\overline{C} \cdot (T * \overline{S})\right)$$

$$T'' = \overline{\mathrm{III}} * \left(\overline{C} \cdot T'\right)$$

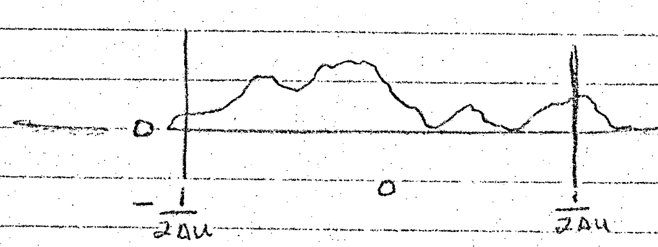
The transform of III is another III function:

$$\overline{\mathrm{III}} = \sum_{-\infty}^{\infty}{}_{m} \sum_{-\infty}^{\infty}{}_{n} \; {}^2\delta\left(x - \frac{m}{\Delta u}, y - \frac{n}{\Delta v}\right)$$

The convolution with $\overline{\mathrm{III}}$ causes the map to repeat at regular intervals like



which is no problem if $\overline{C} \cdot (T * \overline{S})$ looks like the thing above. But if $\overline{C}\; T'$ looks like



then the repetitions will cause real signal ~~to~~ from the "repeated map" to be added in to the central map. Such "folded-in" (an inaccurate terminology) signals are indistinguishable from real signals. Care should be taken to choose Δu and Δv small enough to reduce the "folded-in" signals to less than the noise. If we can ignore the III convolution, we know C to be some exact function (e.g. pill box, gaussian) and hence can obtain

$$T' = \frac{T''}{\bar{C}} = T * \bar{S}$$

It isn't easy to describe what this convolution really means. One property of it is that the total flux on a map may be off:

$$F_{apparent} = \iint dx\,dy\; T' = \iint dx\,dy \iint du\,dv\; V\,S\,e^{-2\pi i(ux+vy)}$$

$$= \iint du\,dv\; V(u,v)\,S(u,v) \iint dx\,dy\; e^{-2\pi i(ux+vy)}$$

$$= \iint du\,dv\; V\,S\;{}^2\delta(u,v)$$

$$= 0 \qquad \text{if no } (u_j, v_j) = (0,0)$$
$$= V(0,0) \qquad \text{if some } (u_j, v_j) = (0,0)$$

Thus, if no zero spacing observation is made, the total flux of the synthesized map is zero.

The following Figures show a real example (Cas A = 3C461). First there is a plot of the set of (u_j, v_j), then contour maps of $\bar{S}$ and T'. The contour maps have contour interval 10% of the peak value and the zero contour is suppressed.

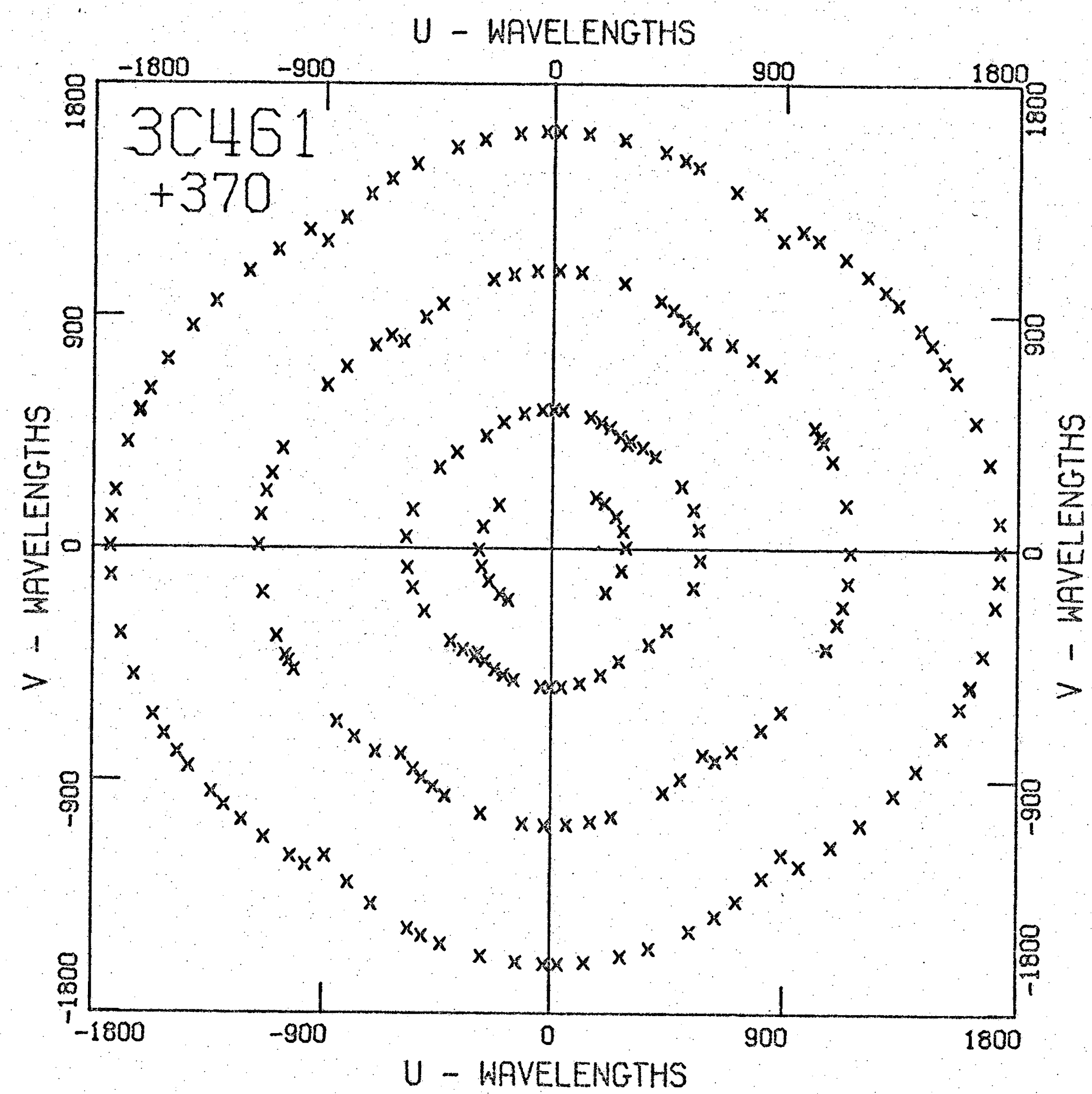


Figure 16. 3C461, continuum frequency shift: distribution of data points.

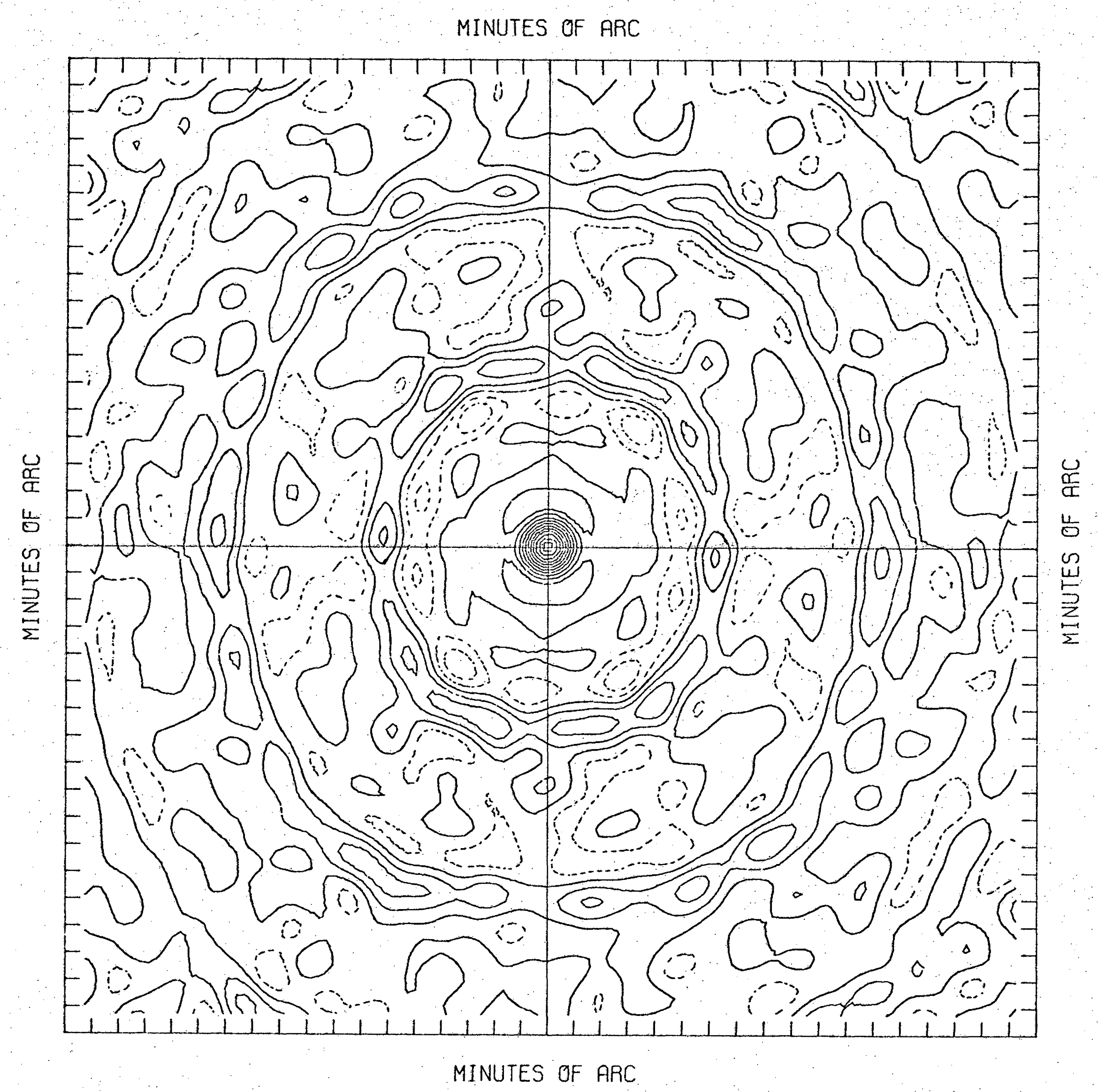
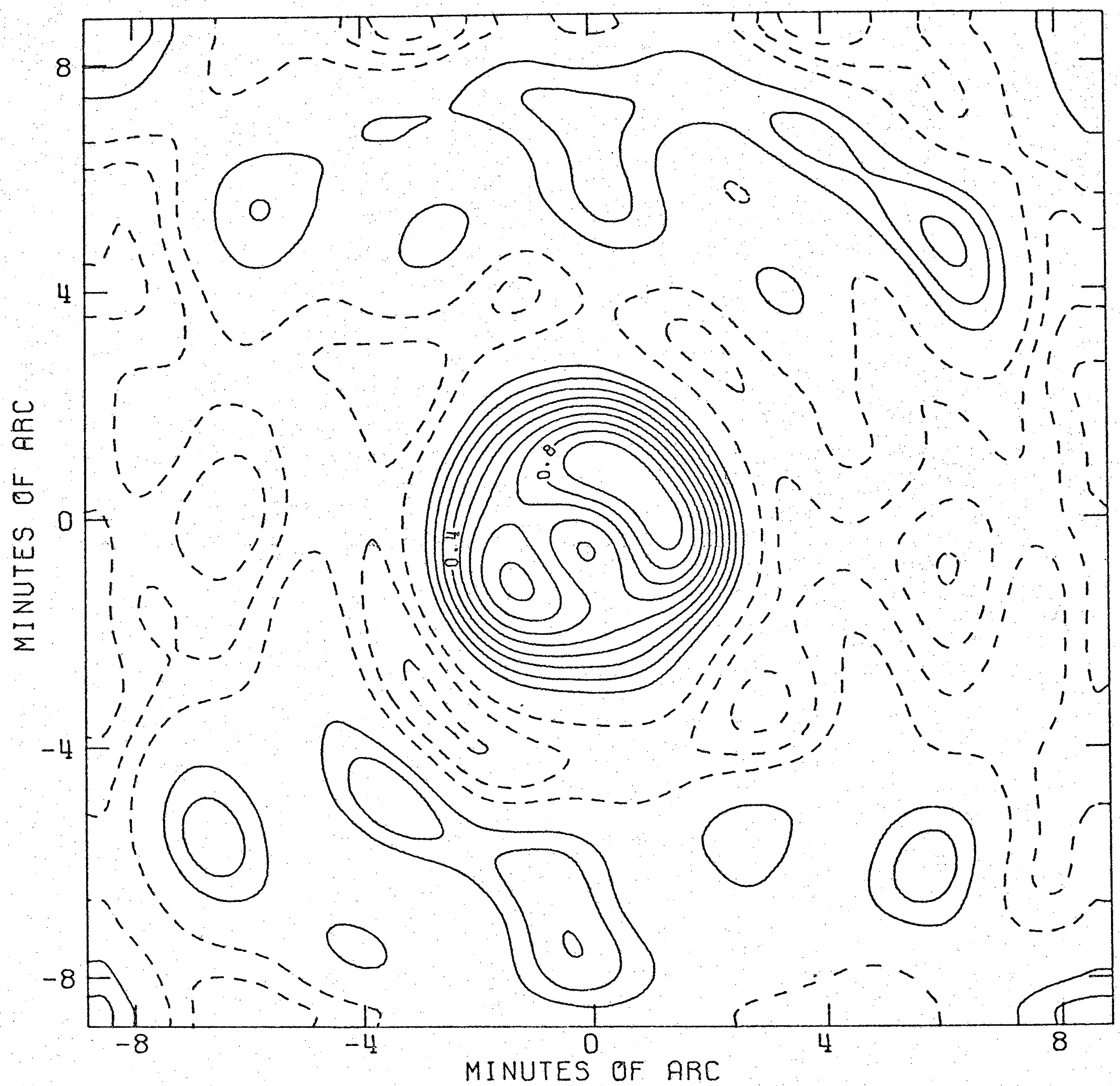


Figure 17a. 3C461, continuum frequency shift: dirty beam pattern, contour interval = 10%, large area.



8
4
0
-4
-8
MINUTES OF ARC
-8
-4
0
4
8
MINUTES OF ARC
0.8
0.4

There have been numerous attempts to perform the deconvolution. One method, called CLEAN is currently popular - at least in some corners. The method goes as follows:

(1) Assume the source consists of an unknown number N of point sources of unknown amplitude and position

$$T(x,y) = \sum T_i \, {}^2\delta(x-x_i, y-y_i)$$

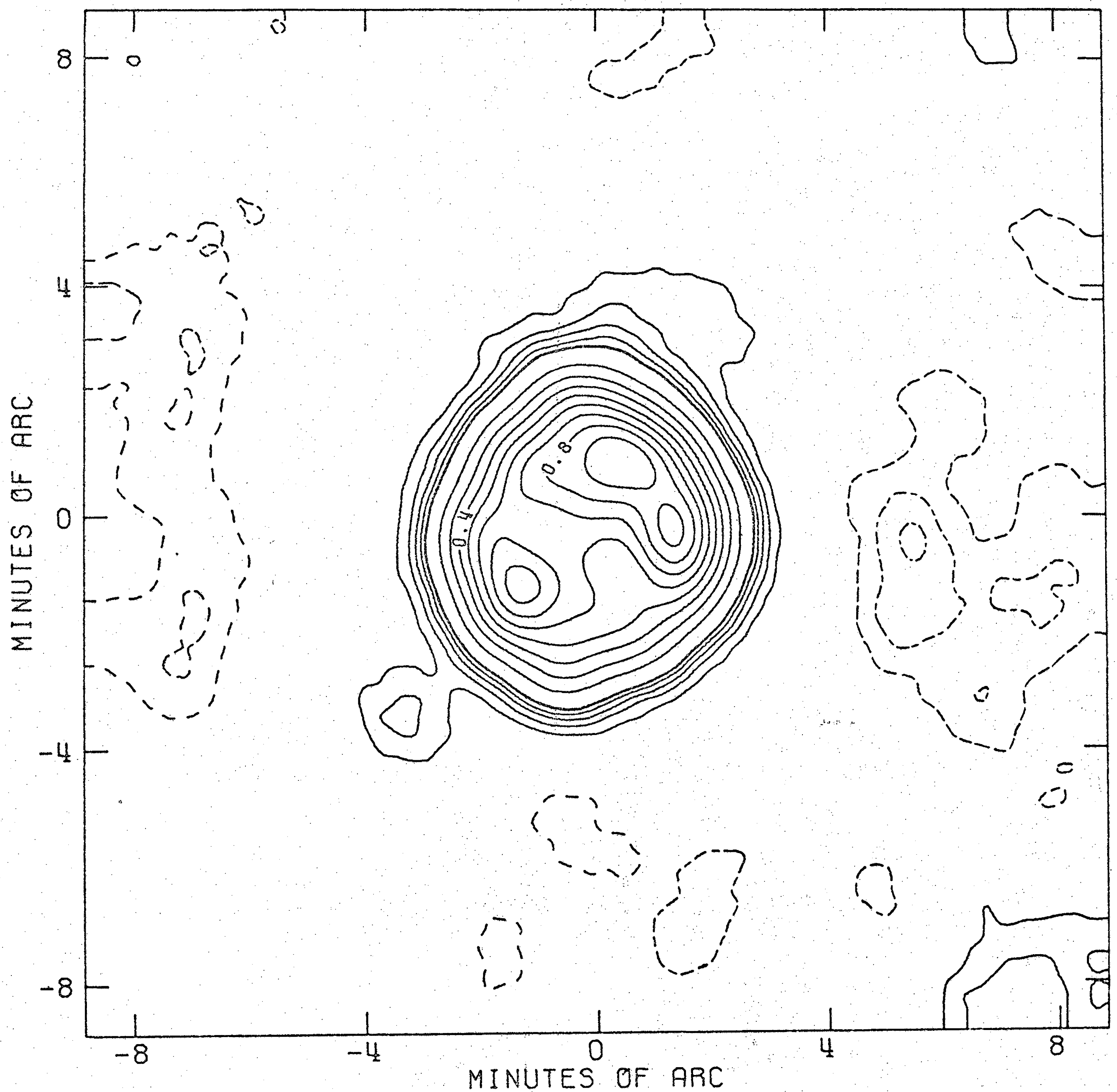
Then

$$T'(x,y) = \sum T_i \, \bar{S}(x-x_i, y-y_i)$$

(2) To find the T_i the program searches the map for the highest peak (in absolute value). Taking this peak as T_1 the program subtracts an appropriately scaled beam pattern centered on the peak from the dirty map. The process is repeated on the residual maps for some number of iterations. Usually astronomers stop the iterative process when $|T_n|$ reaches some noise level.

(3) The last step restores the point sources T_i to the noise map left at the end of step (2). Since the point source model is not likely to be correct, the point sources are convolved with some "clean" beam before being added back to the map. Since the map contains little reliable information on source structures smaller than the central peak of the dirty beam, the clean beam is usually taken as a gaussian having the same diameter as the central peak of the dirty beam.

CLEAN works well on good data - but cannot remove systematic errors, noise, or the effects of extremely poor sampling. The Figure following shows the clean map of Cas A obtained from the previous maps. The contour interval is 10% for contours $> 10\%$ and 2% for contours $< 20\%$ with the 10% contour darkened and the zero contour suppressed.



8
4
0
-4
-8
-8
-4
0
4
8
MINUTES OF ARC
MINUTES OF ARC
0.8
0.4

A real + simple example of interferometry:

In one dimension, take a source consisting of a point source of flux F_1 at $x = -a$ and a point source of flux F_2 at $x = +a$. Let the HI clouds in front of the two sources have optical depths τ_1 and τ_2. Then the visibility function is

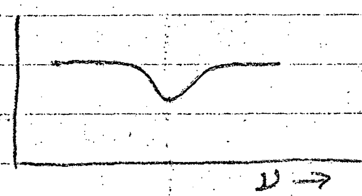
$$V(u,\nu) = F_1 e^{-\tau_1(\nu)} e^{-i 2\pi u a} + F_2 e^{-\tau_2(\nu)} e^{+i 2\pi u a}$$

The maximum of $|V|$ occurs when $u = n/2a$ and the minimum occurs when $u = (n+1/2)/2a$:

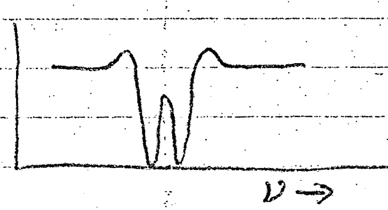
$$|V|_{max} = F_1 e^{-\tau_1} + F_2 e^{-\tau_2}$$
$$|V|_{min} = \left| F_1 e^{-\tau_1} - F_2 e^{-\tau_2} \right|$$

The data show

$|V|_{max}$

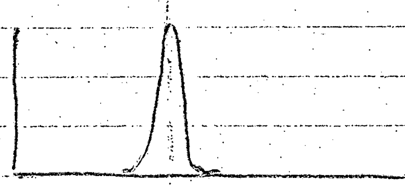


$|V|_{min}$



yielding (with $F_1 > F_2$)

τ_1



τ_2