INTERFEROMETRY

by Eric W. Greisen

for Summer Student Lecture Series 1975

OUTLINE

PART I - To be covered in lecture
- A. Some basic mathematics
- B. A one-dimensional, 2-element interferometer
- C. "Three" dimensions - (u, v) tracks
- D. Mapping
- E. Noise
- F. A simple example

PART II - Not covered in lecture
- A. The real interferometer equations
- B. Some consequences of the equations:
  - i. Baseline measurement
  - ii. Source position measurement - strange coordinates
  - iii. Aperture synthesis
  - iv. Finite bandwidths
  - v. Double sideband systems

References

Lectures, such as this one, which deal with equipment & procedures are known to put audiences to sleep. However, lectures of this sort are important for observers and theoreticians alike. If observational data are to be understood, the systems used to acquire the data must also be understood. Interferometers are already very important instruments in radio astronomy. With the completion of the VLA, they will totally dominate the field.

A. Some basic mathematics needed with interferometers

1. The Fourier transform of a function $f(x,y)$ is given by

$$F(u,v) \equiv \iint_{-\infty}^{\infty} dx\, dy\; f(x,y)\, e^{2\pi i(ux+vy)}$$

where $i = \sqrt{-1}$. The Fourier transform is useful because if the above is true then

$$f(x,y) = \iint_{-\infty}^{\infty} du\, dv\; F(u,v)\, e^{-2\pi i(ux+vy)}$$

is also true.

Demonstration:

$$g(x',y') \equiv \iint du\, dv\; F(u,v)\, e^{-2\pi i(ux'+vy')}$$

$$= \iint dx\, dy \iint du\, dv\; f(x,y)\, e^{+2\pi i[u(x-x')+v(y-y')]}$$

$$= \iint f(x,y)\, dx\, dy \left(\iint_{-\infty}^{\infty} du\, dv\; \cos(2\pi u(x-x'))\cos(2\pi v(y-y')) \right)$$

$$= \iint f(x,y)\; {}^2\delta(x-x', y-y')\, dx\, dy = f(x',y')$$

where ${}^2\delta(\;)$ is a 2-dimensional delta function (see below).

2. Convolution of 2 functions f, g is defined as

$$c(x,y) \equiv f * g = \iint_{-\infty}^{\infty} dx'\, dy'\; f(x',y')\, g(x-x', y-y')$$

Often this operation is a smoothing operation, typically where $f(x,y)$ is a measured function (map) and $g(x,y)$ is a Gaussian.

3. The $^n\delta$ function is an n-dimensional function defined s.t. $\delta = 0$ unless all arguments are zero. When all arguments are zero $\delta = \infty$. The only thing which isn't pathological about δ is its "area"

$$A = \iint_{-\infty}^{\infty} \cdots \int {}^n\delta(x_1, x_2, \ldots, x_n)\, dx_1\, dx_2 \ldots dx_n \equiv 1$$

which is defined to be one.

4. Interesting and useful properties of these operations & functions

a. $f * {}^2\delta(x - x_0, y - y_0)$

$$= \iint dx'dy' \; f(x', y')\; {}^2\delta(x - x' \cancel{+} x_0, y - y' \cancel{+} y_0)$$

$$= f(x - x_0, y - y_0)$$

b. $F.T.\,({}^2\delta(x - x_0, y - y_0))$

$$= \iint dx\, dy \; {}^2\delta(x - x_0, y - y_0)\, e^{2\pi i(ux + vy)}$$

$$= e^{2\pi i(ux_0 + vy_0)}$$

c. $F.T.\,(f * g)$

$$= \iint dx\, dy\, e^{+2\pi i(ux + vy)} \iint dx'dy' \; f(x', y')\, g(x - x', y - y')$$

$$= \iint dx'dy' \; f(x', y') \iint dx\, dy \; g(x - x', y - y')\, e^{2\pi i(ux + vy)}$$

substitute $x'' = x - x'$ $\quad y'' = y - y'$

$$= \iint dx'dy' \; f(x', y') \iint dx''dy'' \; g(x'', y'')\, e^{2\pi i(ux'' + vy'')}\, e^{2\pi i(ux' + vy')}$$

$$= FT(f) \cdot FT(g)$$

d. FT (f · g)

$$= \iint dx\,dy\ f(x,y)\, g(x,y)\, e^{2\pi i(ux+vy)}$$

$$= \iint dx\,dy\ g(x,y)\, e^{2\pi i(ux+vy)} \iint dx'dy'\ f(x',y')\ {}^2\delta(x'-x,\, y'-y)$$

$$= \iint dx\,dy\ g(x,y)\, e^{2\pi i(ux+vy)} \iint dx'dy'\ f(x',y') \iint du'dv'\ e^{2\pi i[u'(x'-x)+v'(y'-y)]}$$

$$= \iint du'dv' \iint dx\,dy\ g(x,y)\, e^{2\pi i[(u-u')x+(v-v')y]} * \iint dx'dy'\ f(x',y')\, e^{2\pi i(u'x'+v'y')}$$

$$= \iint du'dv'\ G(u-u',\, v-v')\, F(u',v')$$

$$= FT(f) \circledast FT(g)$$

where $F \equiv FT(f)$, $G \equiv FT(g)$, and $\circledast$ denotes convolution.

5. Some useful functions:

a. Gaussian:

$$f(x,y) \equiv \exp -\left[+x^2/\sigma_x^2 + y^2/\sigma_y^2\right]$$

$$F(u,v) = \pi \sigma_x \sigma_y \exp -\pi^2\left[\sigma_x^2 u^2 + \sigma_y^2 v^2\right]$$

b. Sha function:

$$III(x,y) \equiv \sum_m \sum_n {}^2\delta(x - m\Delta x,\, y - n\Delta y)$$

$$F.T.(III) = \sum_m \sum_n {}^2\delta(u - m/\Delta x,\, v - n/\Delta y)$$

$$\equiv III(u,v) \quad \text{with } \Delta u = \frac{1}{\Delta x},\ \Delta v = \frac{1}{\Delta y}$$

c. Pill Box (square)

$$f(x,y) = 1/(4x_0 y_0) \qquad |x| < x_0,\ |y| < y_0$$

$$f(x,y) = 0 \qquad |x| > x_0 \text{ or } |y| > y_0$$

$$F(u,v) = \frac{\sin(2\pi u x_0)}{(2\pi u x_0)} \frac{\sin(2\pi v y_0)}{(2\pi v y_0)}$$

d. Uniform disk (circular coordinates)

$$f(r,\theta) = \frac{1}{\pi R_0^2} \qquad r < R_0$$

$$f(r,\theta) = 0 \qquad r > R_0$$

$$F(\rho,\varphi) = \frac{2}{\pi^2} \frac{J_1(2\pi R_0 \rho)}{2\pi R_0 \rho}$$

where $J_1(x)$ is the 1^{st} order Bessel function

B. Let us now consider a very simple interferometer (illustrated on the next page numbered 341 from my thesis). This interferometer consists of 2 telescopes separated by distance D and is shown observing a source at angle θ from the normal. The radio waves from the source are plane, parallel waves because the source is so distant. One of these waves is shown to illustrate that the wave arrives at one telescope τ seconds ahead of the other, where τ is given by

$$\tau = \frac{D}{c} \sin\theta .$$

The voltages produced by the source at each telescope are handled in manner which is complicated and is fully set forth in Part II of these notes. For our purposes here, let V_1 be the voltage at one telescope and V_2 be the voltage at the other. The system produces an output which is the "product" of V_1 and V_2 when the "product" is defined as

$$R = V_1 V_2^* + V_2 V_1^*$$

and where V_2^* is the complex conjugate of V_2, etc. But V_2 is produced by the same wave as V_1 with a time delay. Thus

$$V_2 = V_1 e^{-i\omega\tau}$$

where ω is the observing (angular) frequency. Then

$$R = 2 V_1 V_1^* \cos(\omega\tau)$$

or

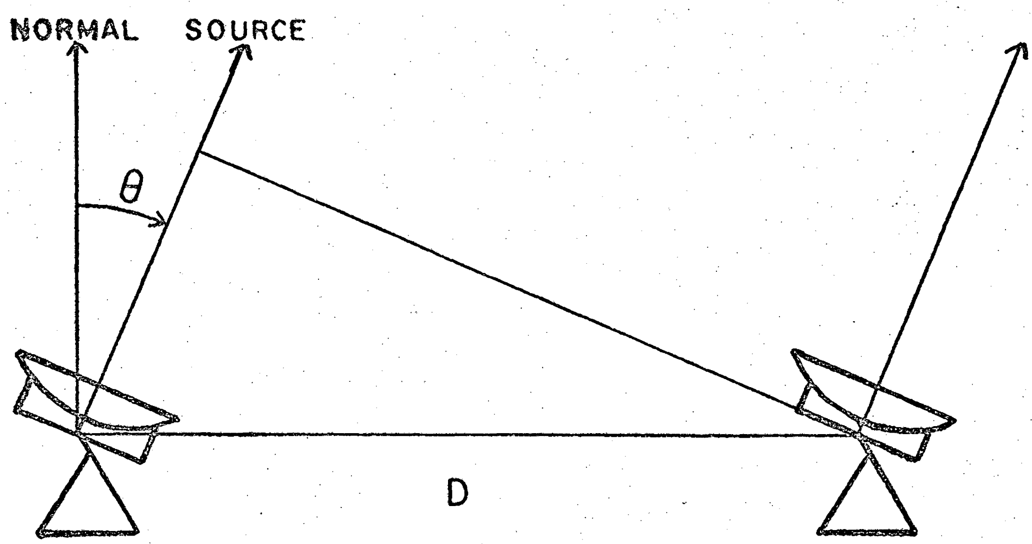

Figure A1. Geometry of two-element interferometer.

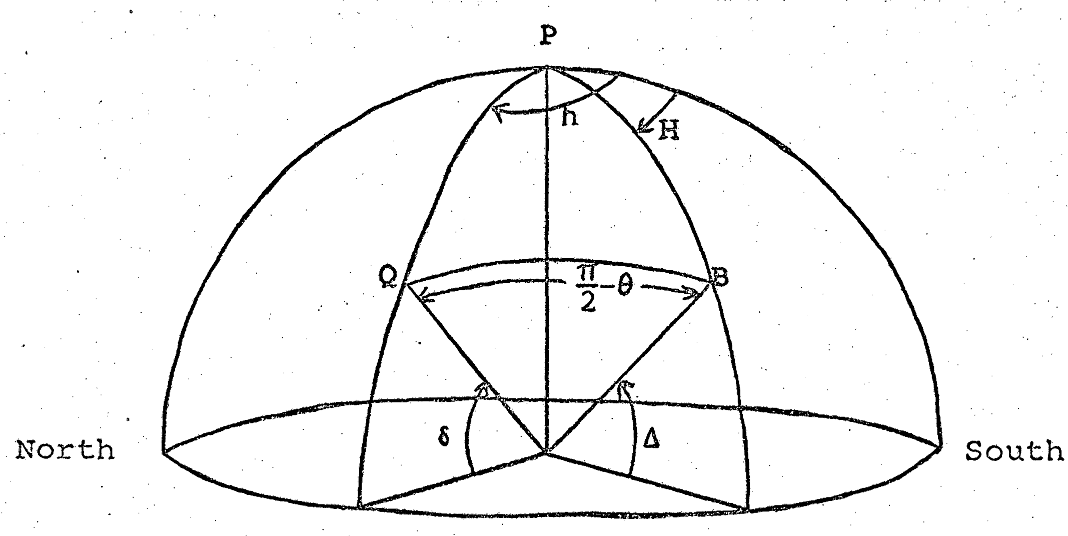

Figure A2. Celestial sphere with source (Q) and pole of baseline (B).

$$R = 2\,G\,T(\theta)\cos\left(2\pi \frac{D}{\lambda}\sin\theta\right) \qquad (B1)$$

where I've substituted the wavelength λ for $\omega/2\pi c$. As the source moves through the sky (i.e. the earth rotates the telescopes around), θ changes producing a sinusoidal output called "fringes". (Note - in equation (B1) above I also substituted for the received power ($2V_1V_1^*$), a different expression showing the receiver gains G and the source power T.)

C. Before we go further with the consequences of (B1) let's generalize the equation to the real word of three dimensions. The half-celestial sphere needed to visualize this generalization is shown in Figure A2 on the previous page. The source Q is shown at hour angle h and declination δ. The projection of the line between the 2 telescopes is shown as point B with hour angle H and declination Δ. We use the spherical trig formula

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

(where the spherical triangle has sides a, b and c and corresponding angles A, B, and C) to obtain

$$\sin\theta = \sin\delta\sin\Delta + \cos\delta\cos\Delta\cos(h-H)$$

or

$$\tau = \frac{D}{c}\left(\sin\delta\sin\Delta + \cos\delta\cos\Delta\cos(h-H)\right)$$

Cal Tech uses baselines expressed in this non-linear fashion, but NRAO prefers a linear method:

z-axis : toward North pole

x-axis : in celestial equator toward the South

y-axis : in celestial equator toward the West

In such a coordinate system, the projections of the baseline vector $\vec{B}$ are (from inspection of Fig A2)

$$B_z = |D| \sin \Delta$$

$$B_x = |D| \cos \Delta \cos H$$

$$B_y = |D| \cos \Delta \sin H$$

Using this coordinate system we find

$$c\tau = B_x \cos \delta \cos h + B_y \cancel{\cos} \, \delta \sin h + B_z \sin \delta . \quad (c1)$$

We can carry this one step further actual, since if $\hat{\sigma}$ is a unit vector in the direction of the source, then

$$\sigma_x = \cos \delta \cos h$$

$$\sigma_y = \cos \delta \sin h$$

$$\sigma_z = \sin \delta$$

and we find

$$\tau = \frac{\vec{B}}{c} \cdot \hat{\sigma} \quad (c2)$$

D. Why do we go to all this trouble? Our response is a precisely known sinusoidal time function whose amplitude has a small amount of information — or is it? So far we've implicitly assumed 2 things — (1) that we know h and δ exactly and (2) that the source has a unique h and δ (i.e. that the source is a point source). However, there are many sources (all?) whose positions aren't known and whose structures aren't known. Let us first relax assumption (1) and suppose that the source is actually at h and δ while we thought it was at h_0 and δ_0. The response is still

$$R = GT \cos\left(\frac{2\pi D}{\lambda} \sin\theta\right)$$

but we expected

$$R_0 = G T_0 \cos\left(2\pi \frac{D}{\lambda} \sin\theta_0\right).$$

In other words the real fringe has some phase relative to the expected fringe. Calling this relative phase the fringe phase, we may express it as

$$\varphi = \frac{2\pi D}{\lambda} \left(\sin\theta - \sin\theta_0\right)$$

or equivalently

$$\varphi = \frac{2\pi}{\lambda} \vec{B}\cdot(\hat{\sigma} - \hat{\sigma}_0).$$

Let's assume our assumptions were pretty good and

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expand $\sin\theta - \sin\theta_0$ to first order in $(\delta-\delta_0)$ and $(h-h_0)$:

Using C1, ~~and~~

$$\Delta(c\tau) = \frac{\partial(c\tau)}{\partial\delta}\Delta\delta + \frac{\partial(c\tau)}{\partial h}\Delta h$$

$$= \Delta\delta\,(-B_x \sin\delta\cos h - B_y \sin\delta\sin h + B_z\cos\delta)$$
$$+(\Delta h\cos\delta)(-B_x\sin h + B_y\cos h)$$

$$= ux + vy$$

where

$$x \equiv \Delta\delta = \delta - \delta_0$$
$$y \equiv \Delta h\cos\delta = (h-h_0)\cos\delta = (\alpha_0 - \alpha)\cos\delta$$
$$u \equiv -B_x\sin h + B_y\cos h$$
$$v \equiv -B_x\sin\delta\cos h - B_y\sin\delta\sin h + B_z\cos\delta$$

form 2 rectangular coordinate systems.

Now, what is our response function like:

$$R = G T_0 \cos\left(\frac{2\pi D}{\lambda}\sin\theta_0 + \varphi\right)$$

$$= G T_0\left[\cos\left(\frac{2\pi D}{\lambda}\sin\theta_0\right)\cos\varphi - \sin\left(\frac{2\pi D}{\lambda}\sin\theta_0\right)\sin\varphi\right]$$

It is easier to express our measurement of amplitude and phase as a complex number:

$$R = G T_0 \exp i\left[\frac{2\pi D}{\lambda}\sin\theta_0 + \varphi\right]$$

When we calibrate and solve for the fringe, we remove G and the expected phase to measure the "fringe visibility" function

$$V = T e^{2\pi i(ux+vy)} \qquad \text{(D1)}$$

Let's now relax assumption (2) and look at an extended source whose power from each infinitesimal solid angle ("brightness temperature") is $T(x,y)$. The resulting calibrated response is just the sum of the infinitesimal responses or

$$V = \iint_{\substack{\text{beam}\\ \text{area}}} T(x,y)\, e^{2\pi i(ux+vy)}\, dx\, dy \qquad \text{(D2)}$$

where the integrals extend over the area to which the single telescopes respond.

Formulae (D1) and (D2) tell us for what an interferometer is good ~~for~~:

(D1) — For a point source of unknown position we may measure fringe phase at several hour angles (e.g. several (u,v) points). Then, we use the above equations to compute $\Delta\delta$ and Δh. Dr Wade talked about such measurements ("astrometry") last week.

(D2) — Our equation says that the fringe visibility function is the Fourier transform of the source brightness distribution. Using the formulae of Part IA (page 2), then we may find the source brightness distribution:

$$T(x,y) = \iint_{-\infty}^{\infty} V(u,v)\, e^{-2\pi i(ux+vy)}\, du\, dv. \qquad (D3)$$

Actually, of course, it isn't that easy. To carry out equation (D3) we must have measured, without noise, the value of V at all (u, v). None of us have an infinite amount of time and the possible baselines are limited ultimately by the size of the Earth and, in practice, by the amount of cable and railroad track we can afford.

Two things help us out:

(1) The hermitian property of V(u, v).

Although V is a complex function, T is real.

Let $V = R + iI$

$$Im(T) = 0 = \iint\left[-R(u,v)\sin 2\pi(ux+vy) + I(u,v)\cos 2\pi(ux+vy)\right] du\, dv$$

Since this is true for x and y:

$$R(u,v) = R(-u,-v)$$
$$I(u,v) = -I(-u,-v)$$

or $\qquad V(u,v) = V^*(-u,-v).$

Thus we can double our data without increasing telescope time.

Actually, this property of V is just a statement of the fact that the only difference between baseline $1\rightarrow 2$ and baseline $2 \rightarrow 1$ is the sign of the phase.

(2) The sampling theorem

Let's assume that we've measured V(u, v) only every Δu in u and Δv in v. This is equivalent to

$$V' = \text{III} \cdot V$$

the transform of which is

$$T' = \overline{\text{III}} * T .$$

The $\overline{\text{III}} = FT(\text{III})$ causes the map of T to repeat at intervals of $1/\Delta u$ in x and $1/\Delta v$ in y. Thus, if the source has the property

$$T(x,y) = 0 \qquad |x| > \frac{1}{2\Delta u}$$
$$\text{or } |y| > \frac{1}{2\Delta v},$$

then $T' = T$ over the area of interest.

Thus we need sample only half the u, v-plane and sample that half at intervals determined by the source dimensions. However, we are restricted only to those baselines available and to the values of (u,v) produced by those baselines.

For a given baseline, the locus of (u,v) points is an ellipse:

$$\frac{u^2}{B_x^2 + B_y^2} + \frac{(v - B_z \cos\delta)^2}{(B_x^2 + B_y^2)\sin^2\delta} = 1 .$$

The actual data points obtained in my thesis are illustrated on the next pages for 3C461 (Cas A $\delta \sim 60°$) and 3C353 ($\delta \sim 0°$). Four east-west and 3 north-south baselines with $-4^{\text{hours}} < h < 4^{\text{hours}}$

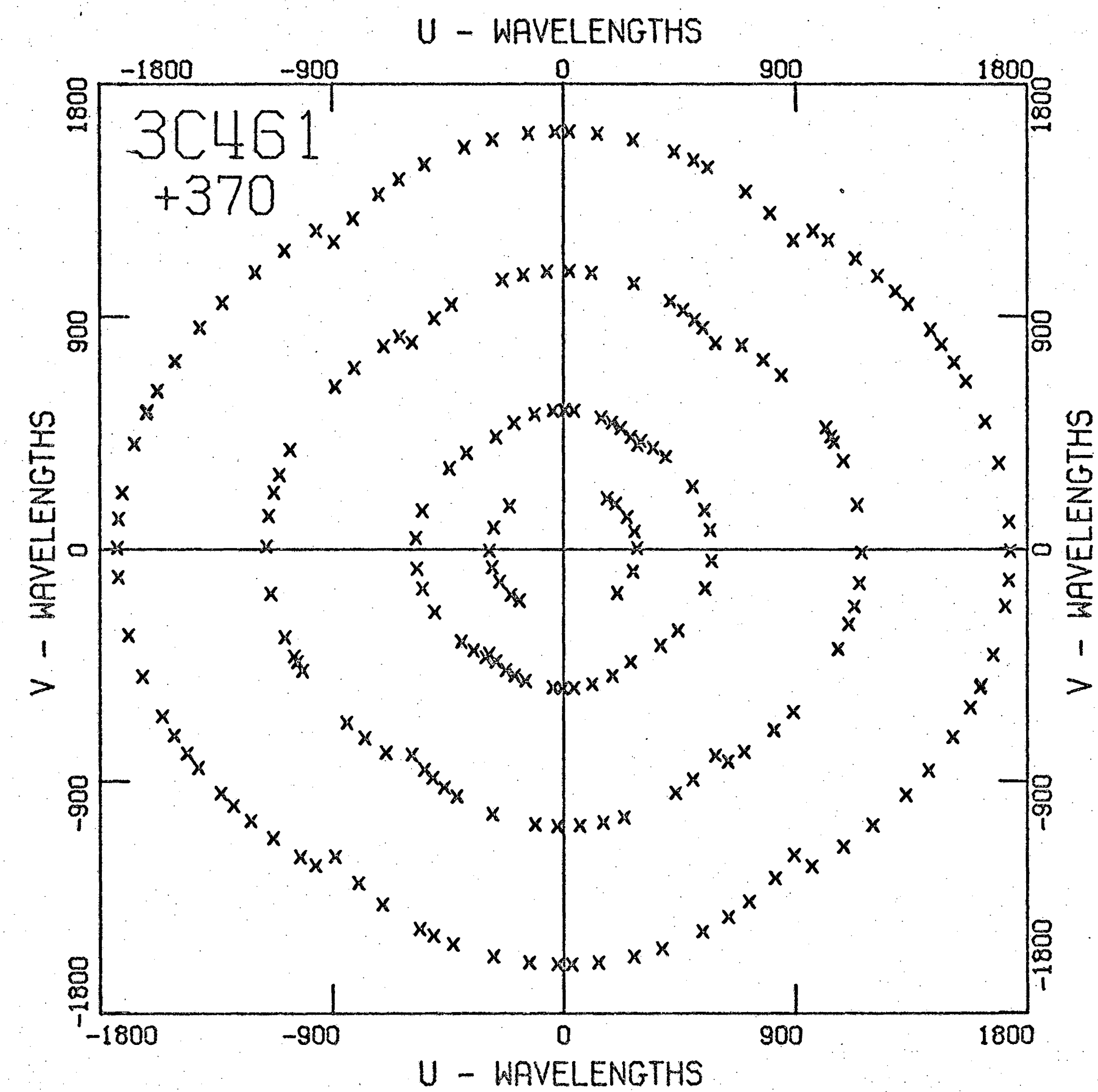


Figure 16. 3C461, continuum frequency shift: distribution of data points.

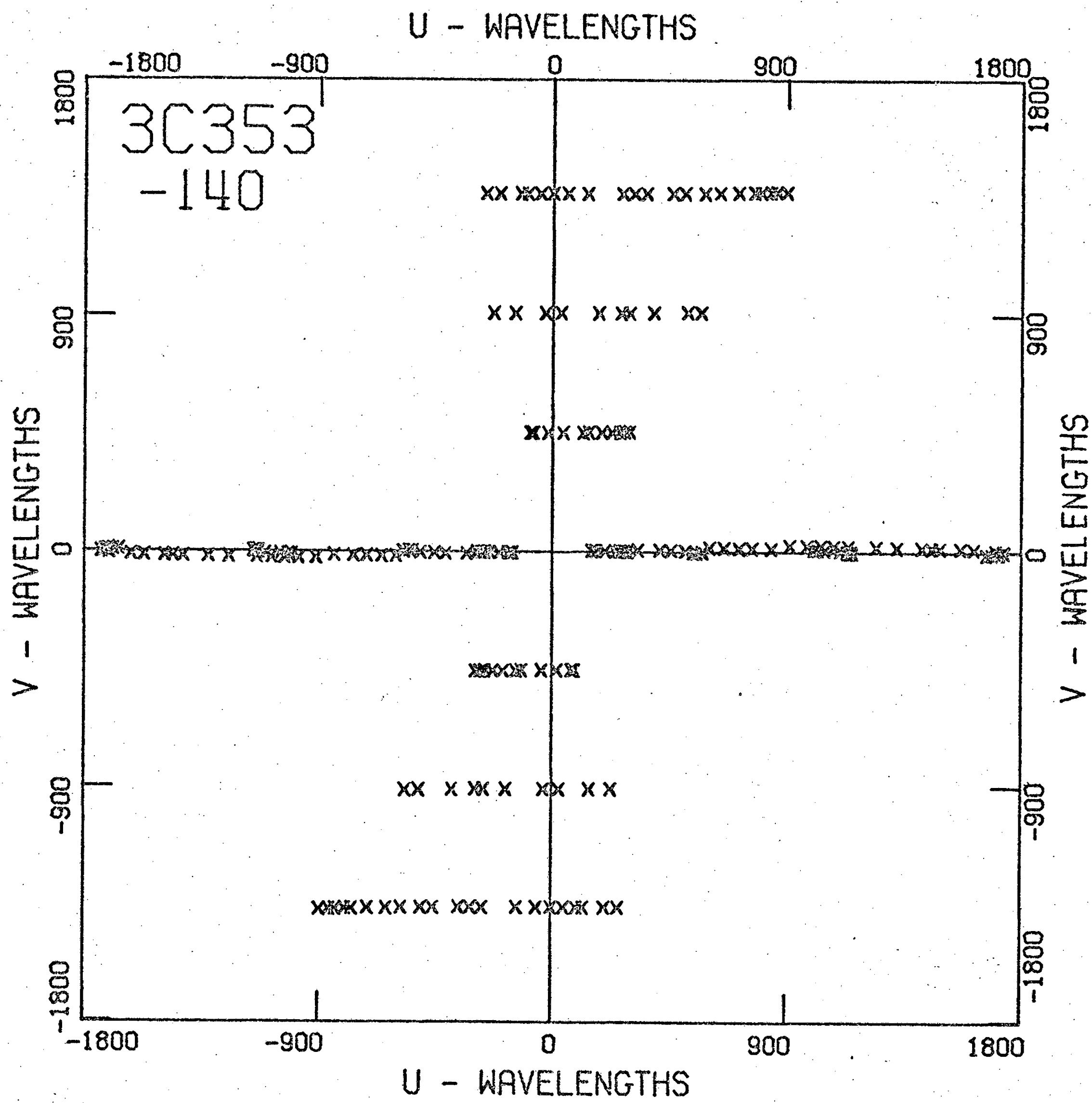


Figure 41. 3C353, continuum frequency shift: distribution of data points.

were used.

What is the effect of this finite set of samples? The visibility function we actually have to transform is not the true one (V) but a sampled one (V')

$$V' = V \cdot S$$

where S is the sampling function

$$S = \sum_i W_i \, {}^2\delta(u-u_i, v-v_i)$$

and the W_i are the weights we assign to the data samples. Then, our resulting map is

$$T' = T * B$$

which is illustrated on the next ~~two~~ 4 pages for the 2 thesis sources. The first of each pair of contour maps is of B ($= FT(S)$) and the second is of T'.

Digression: Some observers like to use the Fast Fourier Transform which requires data samples to lie on a rectangular grid. They get this by smoothing V' and resampling on the regular grid:

$$V'' = III \cdot \left(C * (V \cdot S) \right)$$

where C is a convolving function. Then

$$T'' = \overline{III} * \left(\overline{C} \cdot (T * B) \right).$$

If the $\Delta u, \Delta v$ of III are chosen correctly we may ignore $\overline{III}$ and, since C is a known function, we may recover

$$T^{\flat} = \frac{T''}{\overline{C}} \qquad \text{within} \quad |\Delta x| < \frac{1}{2\Delta u}, \quad |\Delta y| < \frac{1}{2\Delta v}$$

Ignoring $\overline{III}$ is dangerous since, even if T has finite

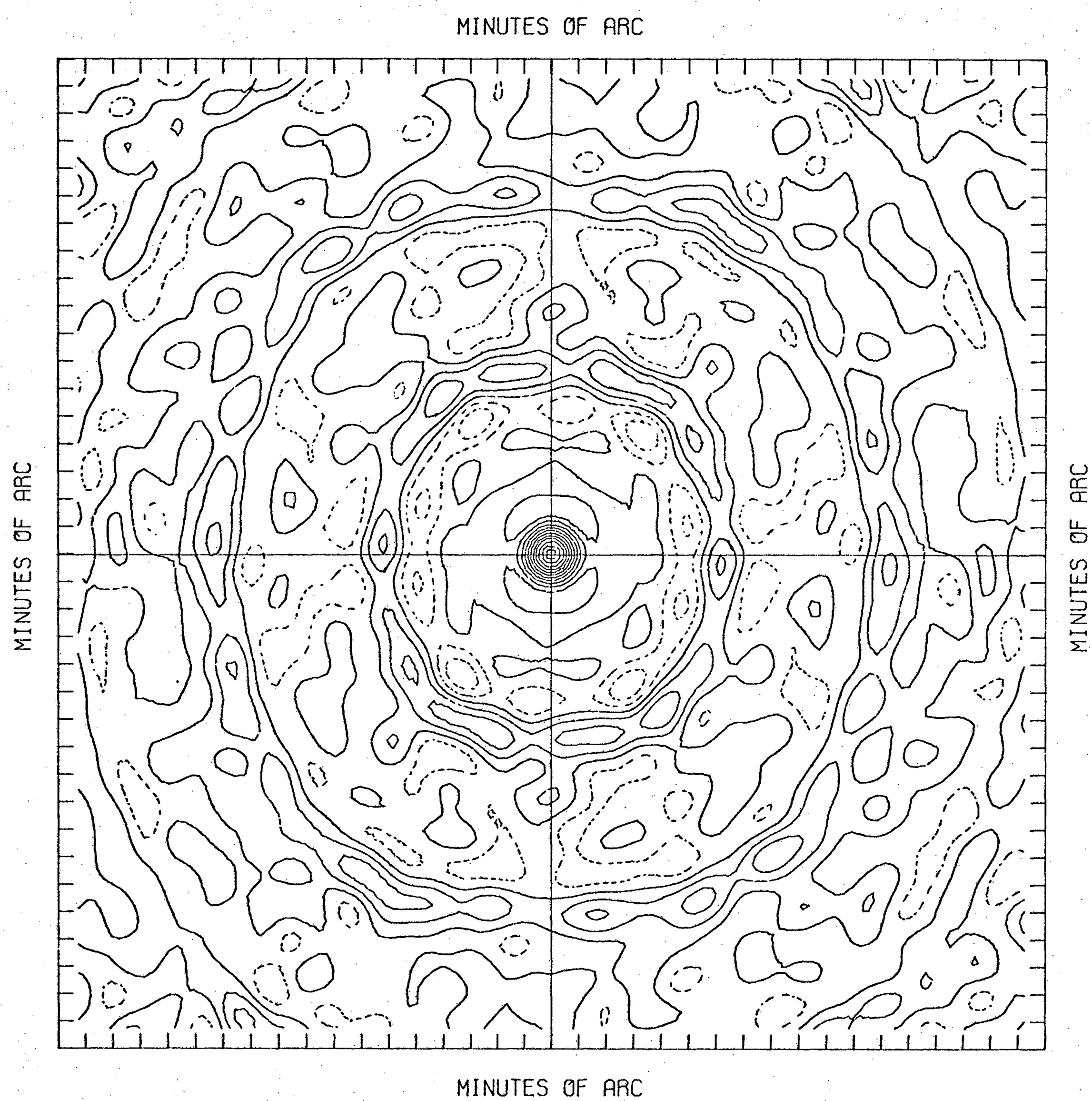


Figure 17a. 3C461, continuum frequency shift: dirty beam pattern, contour interval = 10%, large area.

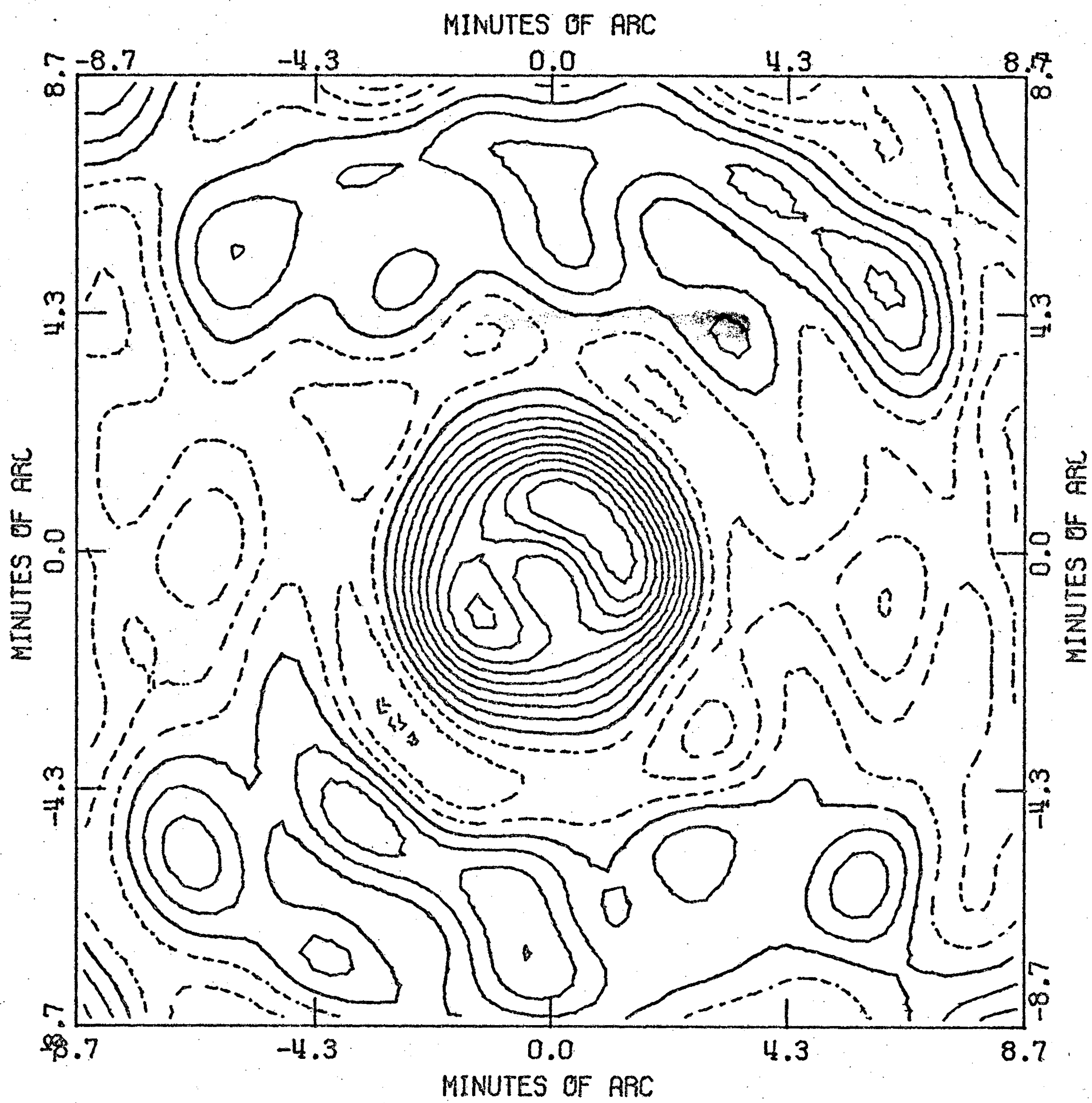


Figure 17b. 3C461, continuum frequency shift: dirty map, contour interval = 10%, large area.

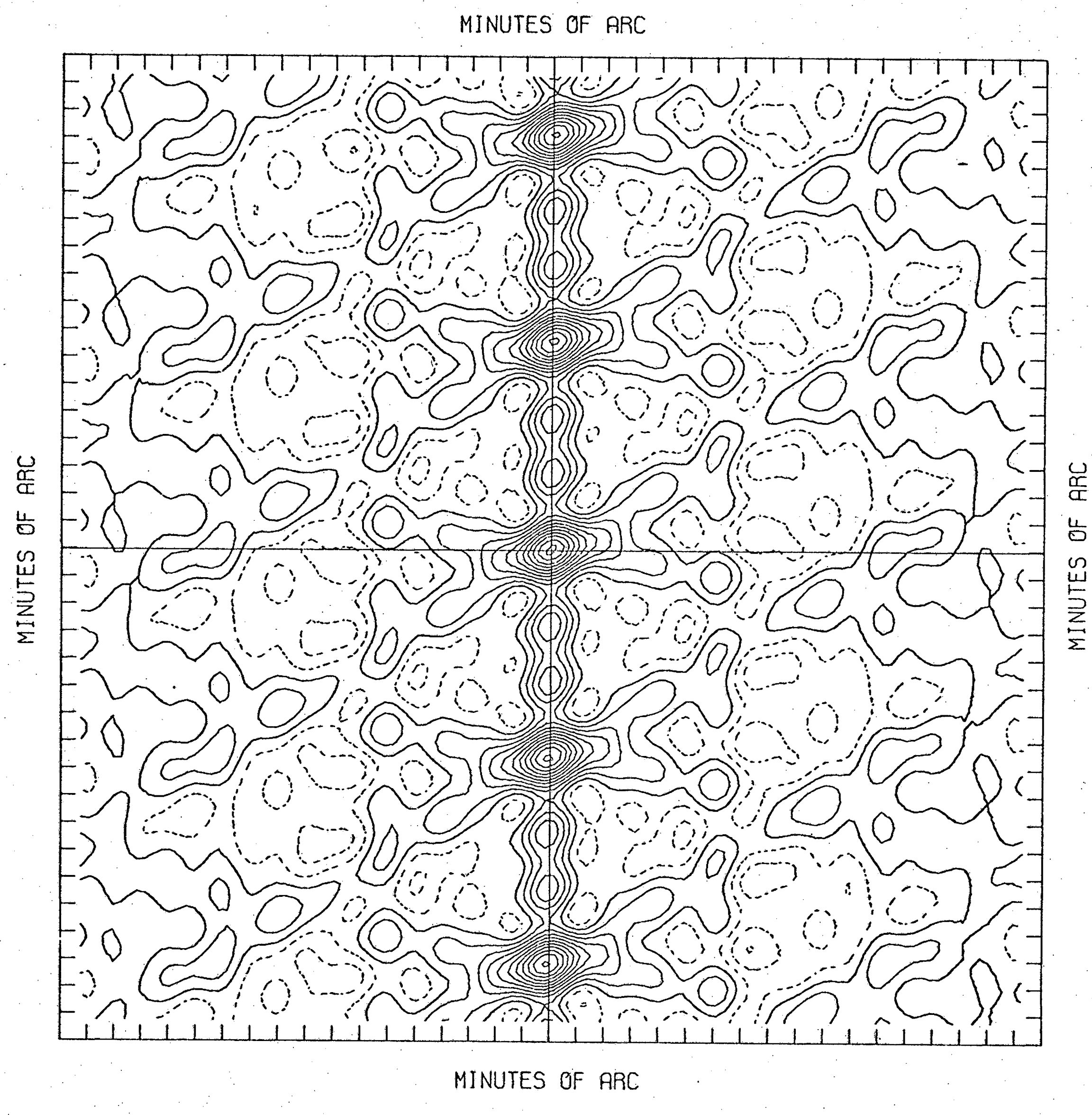


Figure 42a. 3C353, continuum frequency shift: dirty beam pattern, contour interval = 10%, large area.

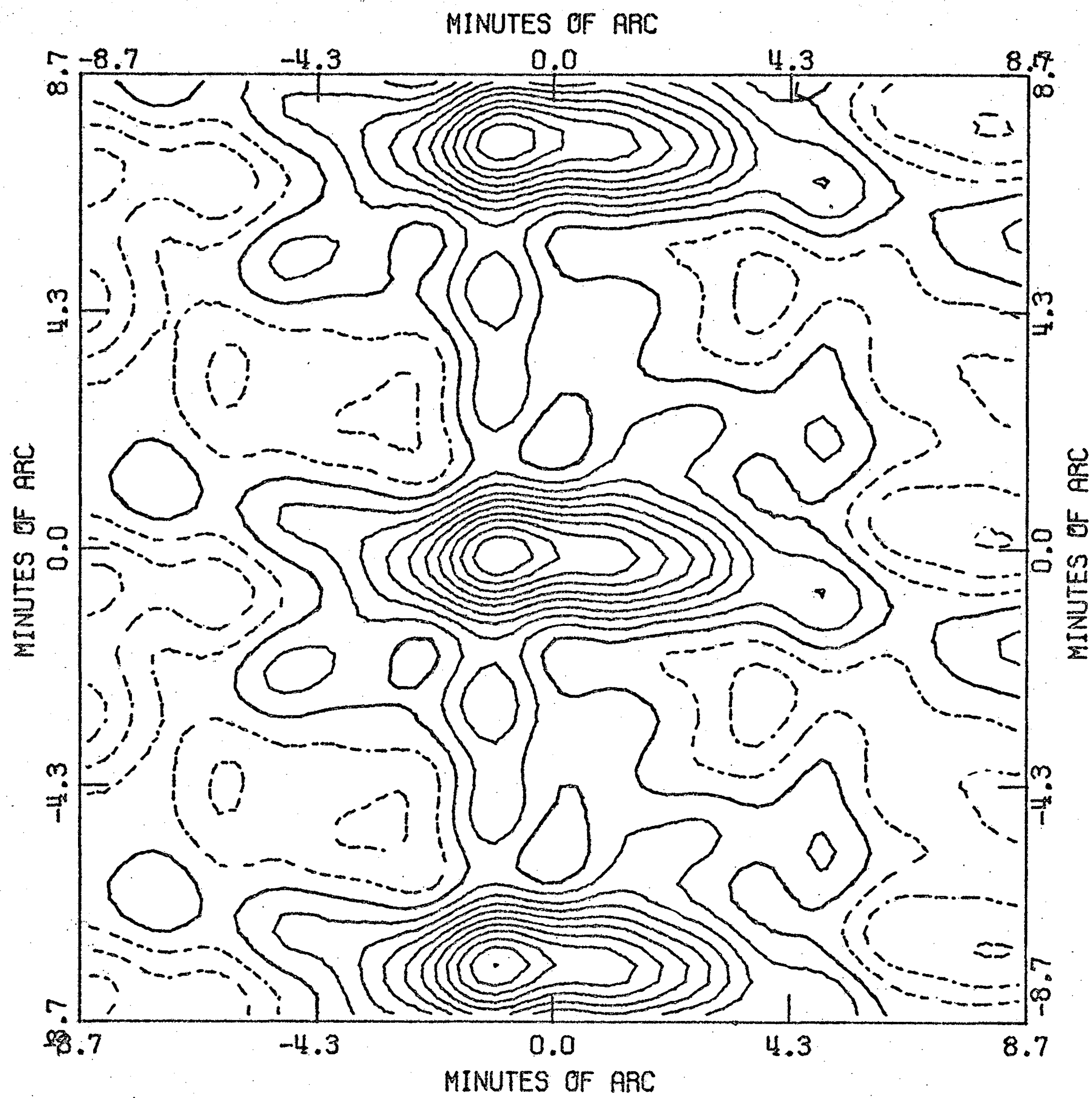


Figure 42b. 3C353, continuum frequency shift: dirty map, contour interval = 10%, large area.

extent, $T * B$ will extend over all (x, y).

The function B is called the "synthesized beam pattern" or "dirty beam" and the function T' is called the "synthesized map" or "dirty map".

There have been many attempts to reduce the effects of the convolution with B. The VLA is one of these. It uses a large number of antennas on a Y shaped set of railroad tracks to try to get nearly uniform sampling of the (u, v) plane. However, until the VLA is completed, we must try other methods. A currently popular one is called CLEAN and consists of several steps:

(1) We assume that the source consists of an unknown number N of components of unknown strength and position but of fixed angular distribution $\equiv M(x, y)$.

e.g. $T(x,y) = \sum_{i=1}^{N} T_i \; M(x-x_i, y-y_i)$

Then

$$T'(x,y) = \sum_{i=1}^{N} T_i \left[\left(M(x,y) * B\right) *^2 \delta(x-x_i, y-y_i)\right]$$

(2) The computer program CLEAN then searches the map for its highest point (which is most likely to be a component). The program then subtracts from the map the response function

$(M(x,y) * B(x,y))$ scaled by the peak temperature found and centered on that peak. The program then searches the remaining map for its highest point and the process is repeated. After some number of iterations, the peak temperature on the residual map is less than the noise (hopefully) and the process stops.

(3) The program (clean) then restores the N components of shape $M(x,y)$ to the residual map, producing what is known as a ~~[illegible]~~ "clean map".

That the process can work well is illustrated for the thesis sources by the contour maps following. However, there are problems. Clean cannot fix up noisy or badly sampled data. Clean may not produce the "correct" answer when the source is not well described by the model distribution $M(x,y)$. Furthermore, the most popular model is the point-source model $M(x,y) = {}^2\delta(x,y)$. Such components have infinite brightness temperature and give contour programs indigestion. People who use this model argue that the smallest source structure their data actually allow is of finite size and hence restore their point sources to the map only after convolving them with a Gaussian "clean beam" of that finite size. This procedure, although very common, is of dubious validity. If CLEAN

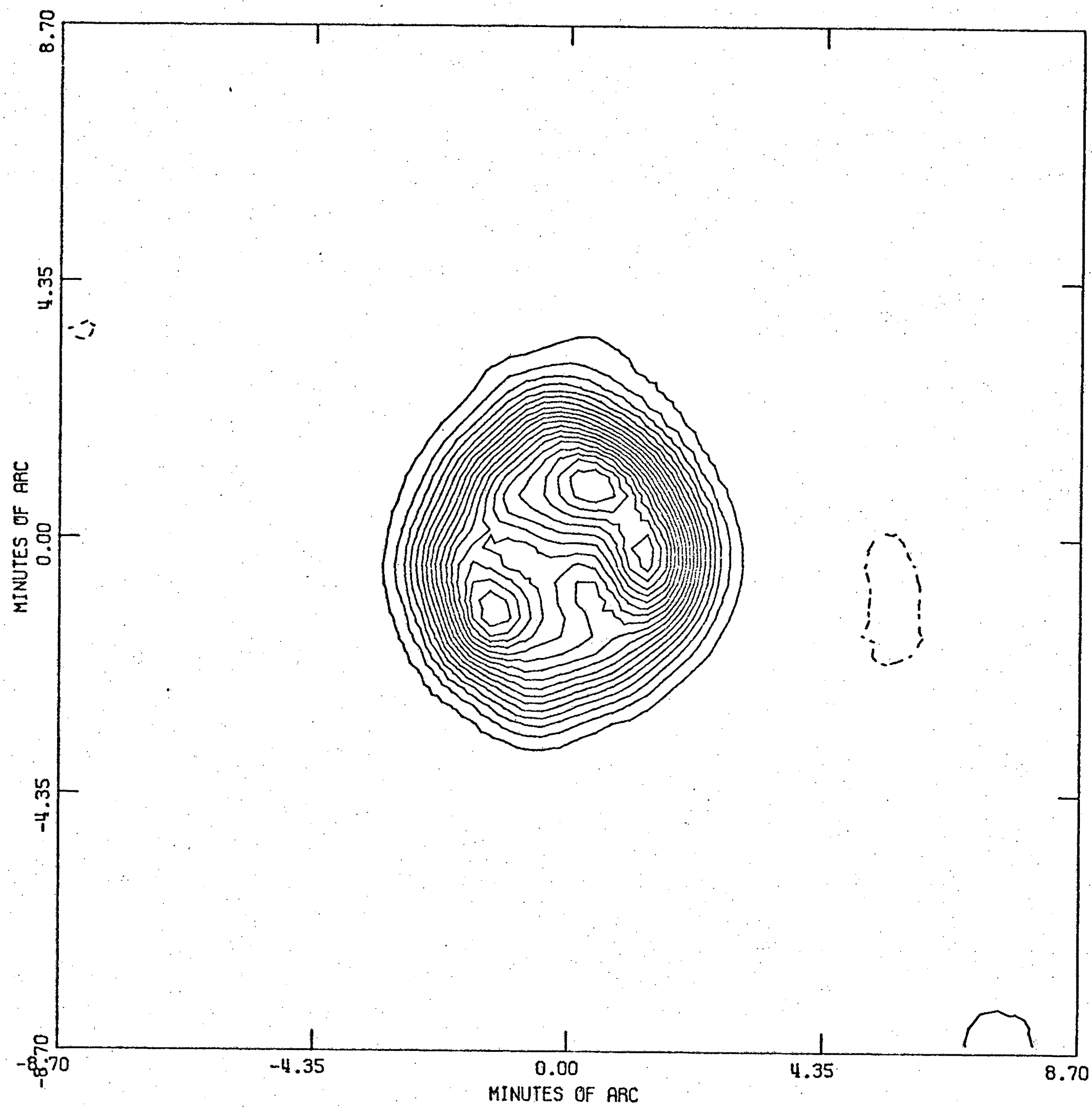


Figure 17c. 3C461, continuum frequency shift: clean map, contour interval = 5%, zero contour suppressed, large area.

Figure 42c. 3C353, continuum frequency shift: clean map, contour interval = 5%, zero contour suppressed, large area.

worked well, the set of point-source components will be a good fit to the actual data.$^{(V')}$ The convolution ~~[illegible]~~ with the clean beam is then equivalent to multiplying V' by the Fourier transform of the clean beam. Other model structures $(M(x,y))$ do not suffer from this problem but may have difficulty fitting sources which actually contain components of angular size less than the angular size of $M(x,y)$.

E. NOISE affects interferometers as much as single dish telescopes. To evaluate the noise on synthesized maps let's carry out the following development. The map itself is found by a Fourier transform which, because the map is real, may be written as

$$T_B(x,y) = K \sum_{j=1}^{N} W_j \, |V_j| \cos(\phi_j + 2\pi(u_j x + v_j y)) \quad (E1)$$

where W_j are the weights assigned the j observations, $|V_j|$ is the fringe amplitude, ~~and~~ ϕ_j the fringe phase, and K a constant (for the moment). Using the Hermitian property

$$T_B(x,y) = \frac{K}{2} \sum_{j=-N}^{N} W_j \, V_j \, e^{-2\pi i (u_j x + v_j y)}$$

where V_j is the complex visibility function. If f is a function of M variables $x_1, x_2, \ldots x_m$ then the uncertainty in f is given by

$$\sigma^2(f) = \left\langle \left| \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \Delta x_i \right|^2 \right\rangle \qquad \langle\ \rangle \Rightarrow \text{expectation}$$

Using this formula and

$$\frac{\partial T_B}{\partial V_j} = \frac{K}{2} \omega_j e^{-2\pi i (u_j x + v_j y)}$$

we obtain

$$\sigma^2(T_B) = \sum_{j=-N}^{N} \sum_{k=-N}^{N} \left(\frac{K}{2}\right)^2 \omega_j \omega_k e^{-2\pi i [(u_j - u_k) x + (v_j - v_k) y]}$$

$$\ast \left\langle \Delta V_j \Delta V_k^* \right\rangle$$

The usual assumption, justifiable in many cases, is that the observations are independent or

$$\langle \Delta V_j \Delta V_k^* \rangle = 0 \qquad (j \neq \pm k)$$

giving

$$\sigma^2(T_B) = \left(\frac{K}{2}\right)^2 \sum_{-N}^{N} \omega_j^2 \left\langle |\Delta V_j|^2 + \Delta V_j \Delta V_j e^{-4\pi i (u_j x + v_j y)} \right\rangle$$

or

$$\sigma^2(T_B) = \frac{K^2}{2} \sum_{1}^{N} \omega_j^2 \left\langle |\Delta V_j|^2 + \mathrm{Re}\left[\Delta V_j \Delta V_j e^{-4\pi i (u_j x + v_j y)}\right] \right\rangle .$$

If we let $V = R + iI$, then the 2^{nd} term contains expressions

$$\left[\langle \Delta R_j^2 \rangle - \langle \Delta I_j^2 \rangle\right] \cos 4\pi (u_j x + v_j y)$$

$$2 \langle \Delta R_j \Delta I_j \rangle \sin 4\pi (u_j x + v_j y) .$$

If receiver noise is the sole source of uncertainty then $\langle \Delta R_j^2 \rangle = \langle \Delta I_j^2 \rangle$ and $\langle \Delta R_j \Delta I_j \rangle = 0$.
If gain or phase calibration uncertainties are important then these nice equalities do not apply, but it is

still found that the contribution of the second term to σ^2 is small. For these reasons, we drop the second term and find that the noise on the map is constant (independent of x & y) and is given by

$$\sigma^2(T_B(x,y)) = \frac{K^2}{2}\sum_{1}^{N} W_j^2 \langle|\Delta V_j|^2\rangle$$

To evaluate this expression further we note:

$$\frac{\langle|\Delta V_j|^2\rangle}{2} = \frac{\langle\Delta R^2\rangle + \langle\Delta I^2\rangle}{2} = \langle\Delta R^2\rangle$$

$$\sqrt{\frac{\langle|\Delta V_j|^2\rangle}{2}} = 2k\,\frac{T_S}{\sqrt{2\,\Delta\nu\, t}}\,\frac{1}{\varepsilon\pi d^2/4}$$

where:
$\Delta\nu$ = bandwidth
t = integration time / sample
ε = antenna aperture efficiency
d = antenna diameter
T_S = effective system temperature (receiver + antenna)
k = Boltzman's constant

and the $\sqrt{2}$ factor arises from having 2 telescopes. To find K let us consider a one flux unit point source at the origin. The Fourier transform gives

$$T_B(0) = K\sum_{1}^{N} W_j$$

but

$$T_B(0) = \frac{\lambda^2}{2k}\frac{\Delta S}{\Delta\Omega} = \frac{\lambda^2}{2k}\frac{\Delta S}{\gamma^2}$$

where ΔS = flux contained in the map cell of size $\Delta\Omega = \gamma^2$ at the origin. We approximate

$$T_B(x,y) * B(x,y) \propto e^{-(4\ln 2)[x^2/\beta_x^2 + y^2/\beta_y^2]}$$

where β_x & β_y are the full widths at half power of the synthesized beam. Then

$$\Delta S = \frac{\iint_{-\gamma/2}^{\gamma/2} B(x,y)\,dx\,dy}{\iint_{-\infty}^{\infty} B(x,y)\,dx\,dy}$$

$$= \left[\mathrm{erf}\left(\frac{\gamma\sqrt{\ln 2}}{\beta_x}\right)\mathrm{erf}\left(\frac{\gamma\sqrt{\ln 2}}{\beta_y}\right)\right] \equiv \frac{\gamma^2}{\beta_x\beta_y}\quad G(\gamma,\beta_x,\beta_y)$$

where G is a very slowly varying function ~ 0.85 for normal γ/β ratios. Then

$$K = \frac{T_0}{\Sigma w_i} = \left(\frac{\lambda^2}{\beta_x\beta_y}\right)\frac{1}{2k}\;\frac{G(\gamma,\beta_x,\beta_y)}{\Sigma w_i}$$

or, substituting

$$\sigma(T_B) = \frac{2^{3/2}}{\pi}\;\frac{G(\gamma,\beta_x,\beta_y)}{\varepsilon d^2}\left(\frac{\lambda^2}{\beta_x\beta_y}\right)\frac{T_S}{\sqrt{\Delta\nu\, T}}\;\frac{1}{E} \qquad E2$$

where $E^2 = \dfrac{(\overline{W})^2}{(\overline{W^2})}$ and $\overline{}$ represents average.

and $T = Nt$ = total integration time.

Thus, as resolution improves in aperture synthesis observations, the noise gets worse (so long as T and E remain constant). This is a quite reasonable result since improved resolution means that we've synthesized a larger aperture but have kept the amount of that aperture actually filled by the 2 small telescopes constant (e.g. T constant). Noting that

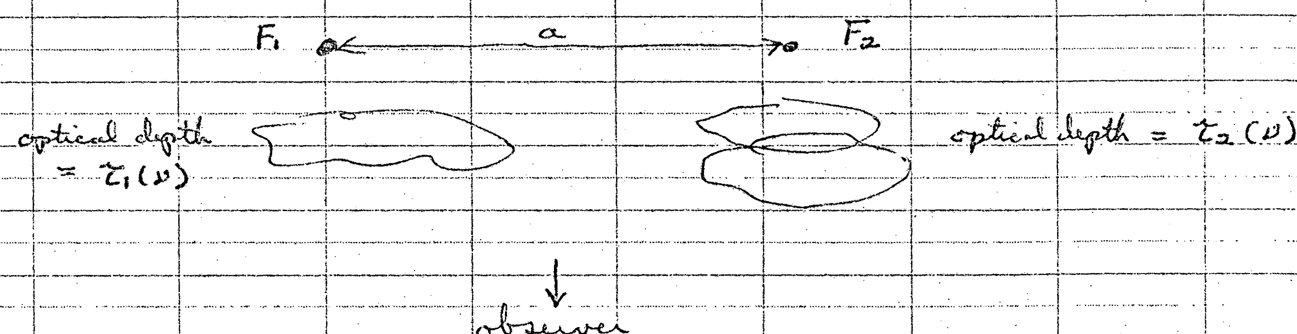
$$\frac{\lambda^2}{d^2} \propto \left(\beta_{\text{single dish}}\right)^2$$

we can write

$$\sigma_{\text{INT.}}(T_B) \propto \left(\frac{\text{single-dish beam area}}{\text{synthesized beam area}}\right) \sigma_{SD}(T_B)$$

where $\sigma_{SD}(T)$ is the uncertainty in T_B of a single dish observation with bandwidth $\Delta\nu$ and integration time T. The proportionality factor will only be some small correction (~ 1.0) for various efficiencies.

F. To conclude Part I let's look at a very simple, but also real, example. Let's consider a source consisting of 2 point components of fluxes F_1 and F_2. Let us also put between us and the source some clouds of neutral hydrogen. This hydrogen will partially absorb the emission from the components at those frequencies corresponding to the Doppler velocities of the hydrogen. Our picture looks like:



and our observer has a many-channel interferometer capable of measuring separately the apparent source visibility functions at many closely spaced values of ν.

The visibility function is given by

$$V_\nu(u,v) = F_1 e^{-\tau_1(\nu)} e^{2\pi i(ux_1+vy_1)\nu} + F_2 e^{-\tau_2(\nu)} e^{2\pi i(ux_2+vy_2)\nu}$$

We examine the amplitude of V at two spots:

(1) $u = v = 0 \qquad |V| = \left| F_1 e^{-\tau_1} + F_2 e^{-\tau_2} \right|$

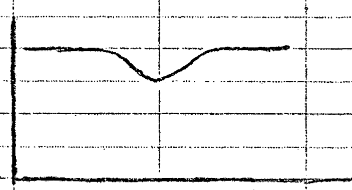
(2) u, v s.t. $|V(\tau=0)|$ is a minimum

$$|V| = \sqrt{(F_1 e^{-\tau_1})^2 + (F_2 e^{-\tau_2})^2 + 2 F_1 e^{-\tau_1} F_2 e^{-\tau_2} * \cos\left[2\pi\left(u(x_1-x_2)+v(y_1-y_2)\right)\right]}$$

$$|V|_{\substack{\tau=0 \\ min}} = \left| F_1 e^{-\tau_1} - F_2 e^{-\tau_2} \right|$$

Suppose we know that $F_1 > F_2$ and that the data show

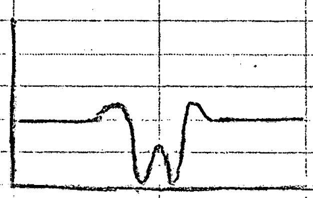
$|V|_{max}$



$\nu \rightarrow$

$F_1 e^{-\tau_1} + F_2 e^{-\tau_2}$

$|V|_{min}$

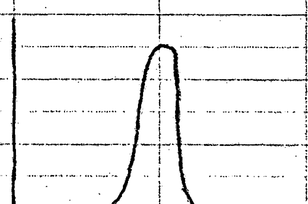


$\nu \rightarrow$

$\left| F_1 e^{-\tau_1} - F_2 e^{-\tau_2} \right|$

Then we may deduce

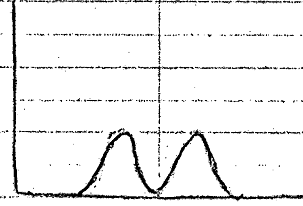
τ_1



$\nu \rightarrow$

(See my paper referenced on next page for the actual data).

τ_2



$\nu \rightarrow$

I. References

A. INTERFEROMETRY

a. Hjellming, R.M. 1973, *An Introduction to the NRAO Interferometer* - chapters 1 and 2 of this manual are of interest to non-users

b. Balick, B. 1973, *Interferometry and Aperture Synthesis*, unpublished thesis & summer student lecture notes (1973) - a good pictoral approach but contains a few errors

c. Fomalont, E.B. 1973, "Earth Rotation Aperture Synthesis", *Proc IEEE*, 61, 1211 - available in GB and Ivy Rd libraries

d. Read, R.B. 1963, *Ap. J.*, 138, 1. - basic reference on interferometer phase

B. SAMPLES OF THE LITERATURE

a. Hogg, D.E. *et al.* 1969, *A. J.*, 74, 1206 continuum interferometry

b. Greisen, E.W. 1973, *Ap. J.*, 184, 363 and 379 line interferometry

c. Weiler, K.W. *et al.* 1971, *Ap. J.*, 163, 455 polarization interferometry

PART II

How a real interferometer works:

A real interferometer system is quite complicated. For reference, edification, and such good things I include, in these notes only, a treatment of such a real system. The system is sketched in Figure A-1 and we define the following symbols:

x_1, x_2, y_1, y_2, z	electrical lengths of cables in telescope 1
$x_1+\Delta x_1,\ x_2+\Delta x_2,\ y_1+\Delta y_1,\ y_2+\Delta y_2,\ z+\Delta z$	corresponding electrical lengths of cables in telescope 2
ω_{LO}	angular frequency of 1st local oscillator
ω_2	angular frequency of 2nd local oscillator
ω_a	angular frequency of 1st lobe rotator
ω_b	angular frequency of 2nd lobe rotator
θ	phase shift added by 1st lobe rotator
α	phase shift added by 2nd lobe rotator
ω_{IF}	angular frequencies to which amplifiers following the delay lines (Δz) respond
$A(\beta,\gamma)$	voltage produced by element of source at coordinates β,γ with telescope 1
$B(\beta,\gamma)$	as A but for telescope 2

To distinguish frequencies $> \omega_{LO}$ we will use subscript u for upper sideband and L (lower sideband) for frequencies $< \omega_{LO}$.

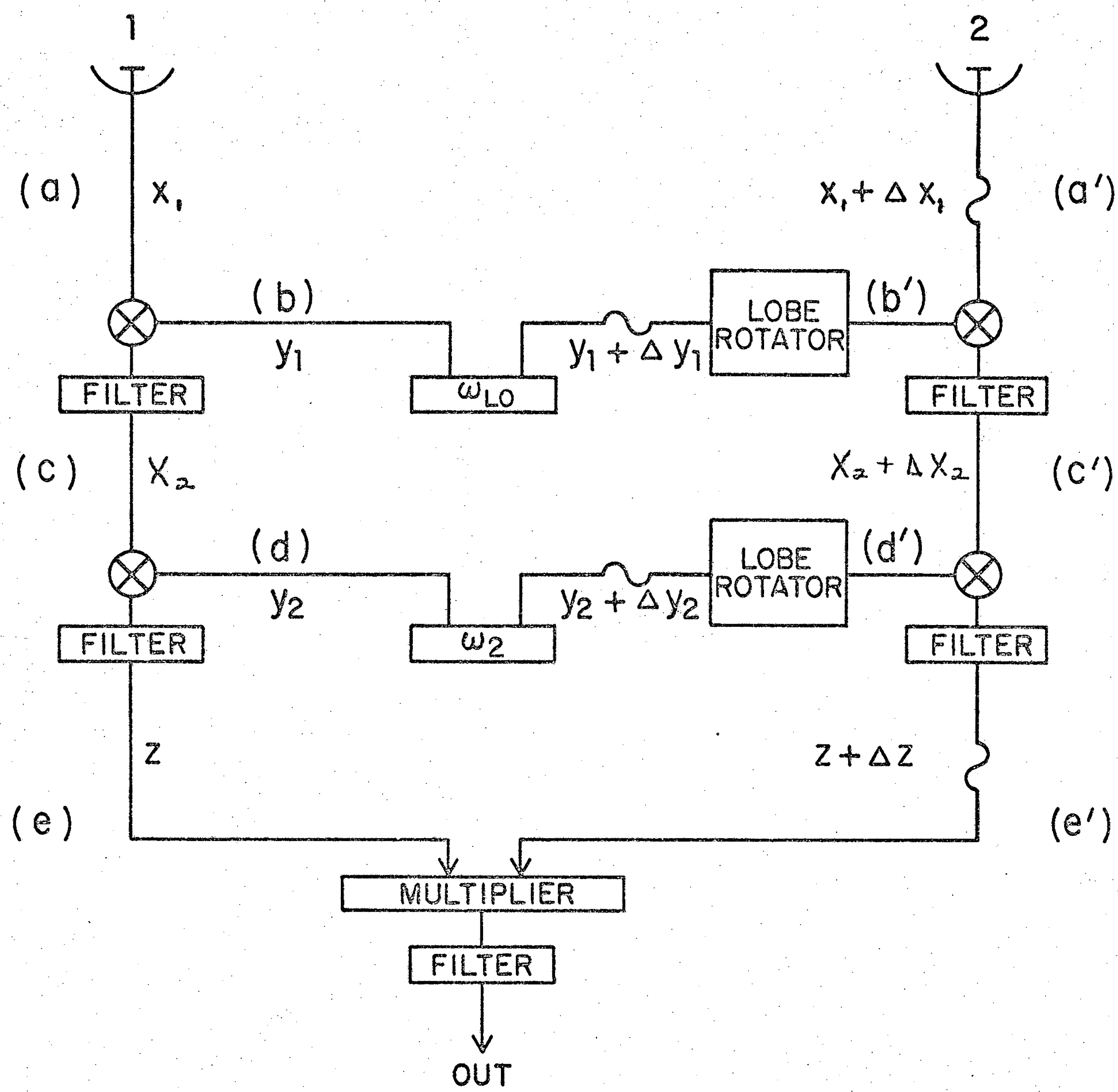


Figure A-1. Receiver Logic

Before beginning the analysis, let's note that A and B are random signals. This means that the time average of $A \cdot B$ is given by

$$< A(\beta, \gamma, \omega, t)\ B(\beta', \gamma', \omega', t') >_{\text{average over } t}$$

$$= G(\omega)\ T(\beta, \gamma, \omega)\ {}^4\delta(\omega - \omega', \beta - \beta', \gamma - \gamma', t - t')$$

where G is some gain function

$T(\beta, \gamma, \omega)$ is the source brightness temperature

${}^4\delta(\quad)$ is a 4-dimensional delta function which is zero everywhere except when all arguments are zero.

Let us now follow the voltages in the system for telescope 1. They are given by the real parts of

at (a) $V = A_u \exp i[(\omega_{LO} + \omega_2 + \omega_{IF})(t - x/c)]$
$\quad + A_L \exp i[(\omega_{LO} - \omega_2 - \omega_{IF})(t - x/c)]$

at (b) just before the first mixer:

$$L = \exp i[\omega_{LO}(t - y_1/c)]$$

The mixer multiplies the signals at (a) and (b) producing frequencies of $2\omega_{LO} \pm \omega_2 \pm \omega_{IF}$ and frequencies $\pm\omega_2 \pm \omega_{IF}$. The filter cuts off the high ($2\omega_{LO}$) frequencies. It may be shown that a mixer outputs, after such filtering,

$$V_u L^* + V_L^* L \qquad (1)$$

where the * denotes complex conjugate. Thus at (c)

$$V = A_u \exp i[-\omega_{LO}(x_1/c - y_1/c) + (\omega_2+\omega_{IF})(t - x_1/c)]$$
$$+ A_L^* \exp i[\omega_{LO}(x_1/c - y_1/c) + (\omega_2+\omega_{IF})(t - x_1/c)]$$

Just before the next mixer we have

$$V = A_u \exp i[-\omega_{LO}(x_1/c - y_1/c) + (\omega_2+\omega_{IF})(t - x_1/c - x_2/c)]$$
$$+ A_L^* \exp i[\omega_{LO}(x_1/c - y_1/c) + (\omega_2+\omega_{IF})(t - x_1/c - x_2/c)]$$
$$L' = \exp i[\omega_2(t - y_2/c)]$$

We must now point out another action of the previous filter — or in real systems additional local oscillators — to restrict ω_{IF} to $\omega_{IF} > 0$. Thus the second mixer only has an upper sideband and we obtain at (e)

$$V_A = A_u \exp i[-\omega_{LO}(x_1/c - y_1/c) - \omega_2(x_1/c + x_2/c - y_2/c)$$
$$+\omega_{IF}(t - x_1/c - x_2/c - z/c)]$$
$$+ A_L^* \exp i[\omega_{LO}(x_1/c - y_1/c) - \omega_2(x_1/c + x_2/c - y_2/c)$$
$$+\omega_{IF}(t - x_1/c - x_2/c - z/c)]$$

<u>NOTE</u> : For simplicity I have not shown the integrals over β, γ, and ω_{IF}. Throughout this discussion, we must remember that they are present.

Now, for telescope 2, the signal from (β, γ) is delayed by $\tau(\beta,\gamma)$ seconds compared to telescope 1. Thus at (a')

$$V = B_u \exp i[(\omega_{LO}+\omega_2+\omega_{IF})(t - \tau - x_1/c - \Delta x_1/c)]$$
$$+ B_L \exp i[(\omega_{LO}-\omega_2-\omega_{IF})(t - \tau - x_1/c - \Delta x_1/c)]$$

To drive the lobe rotators, we must assume a source position and know (or assume) the lengths of the baselines. Such assumptions will be indicated by the subscript 0.

We drive lobe rotator 1 at phase $-\omega_a \tau_0 + \theta$
giving at (b')

$$L = \exp i\left[\omega_{LO}(t - y_1/c - \Delta y_1/c) - \omega_a \tau_0 + \theta\right]$$

and at (c')

$$\begin{aligned} V = {} & B_u \exp i\left[-\theta + \omega_a \tau_0 - \omega_{LO}\tau - \omega_{LO}(x_1/c + \Delta x_1/c - y_1/c - \Delta y_1/c) \right. \\ & \left. + (\omega_2 + \omega_{IF})(t - \tau - x_1/c - \Delta x_1/c - x_2/c - \Delta x_2/c)\right] \\ & + B_L^* \exp i\left[\theta - \omega_a \tau_0 + \omega_{LO}\tau + \omega_{LO}(x_1/c + \Delta x_1/c - y_1/c - \Delta y_1/c) \right. \\ & \left. + (\omega_2 + \omega_{IF})(t - \tau - x_1/c - \Delta x_1/c - x_2/c - \Delta x_2/c)\right] \end{aligned}$$

Lobe rotator 2 shifts phase by $-\omega_b \tau_0 + \alpha$ yielding
at (d')

$$L = \exp i\left[\omega_2(t - y_2/c - \Delta y_2/c) + \alpha - \omega_b \tau_0\right]$$

at (e')

$$\begin{aligned} V_B = {} & B_u \exp i\left[-\theta - \alpha + (\omega_a + \omega_b)\tau_0 - (\omega_{LO} + \omega_2)\tau \right. \\ & - \omega_{LO}(x_1/c + \Delta x_1/c - y_1/c - \Delta y_1/c) \\ & - \omega_2(x_1/c + \Delta x_1/c + x_2/c + \Delta x_2/c - y_2/c - \Delta y_2/c) \\ & \left. + \omega_{IF}(t - \tau - x_1/c - \Delta x_1/c - x_2/c - \Delta x_2/c - z/c - \Delta z/c)\right] \\ & + B_L^* \exp i\left[\theta - \alpha - (\omega_a - \omega_b)\tau_0 + (\omega_{LO} - \omega_2)\tau \right. \\ & + \omega_{LO}(x_1/c + \Delta x_1/c - y_1/c - \Delta y_1/c) \\ & - \omega_2(x_1/c + \Delta x_1/c + x_2/c + \Delta x_2/c - y_2/c - \Delta y_2/c) \\ & \left. + \omega_{IF}(t - \tau - x_1/c - \Delta x_1/c - x_2/c - \Delta x_2/c - z/c - \Delta z/c)\right] \end{aligned}$$

The output of the system is given by

$$V_A V_B^* + V_A^* V_B$$

after terms in $2\omega_{IF} t$ have been filtered by the low-pass filter.
Taking advantage of the fact that the A's and B's are
random voltages (see p 3), we can write the result as

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$$R_r = \int_0^\infty d\omega_{IF} \iint_{\substack{\text{antenna}\\\text{beam}}} d\beta\, d\gamma \Big[G_u(\omega_u) T_u(\beta,\gamma,\omega_u) \cos(\Phi_u + \theta + \alpha) + G_L(\omega_L) T_L(\beta,\gamma,\omega_L) \cos(-\Phi_L - \theta + \alpha) \Big] \quad (2)$$

where

$$\Phi_{\substack{u\\L}} = (\omega_{LO} \pm \omega_2 \pm \omega_{IF})\,\tau(\beta,\gamma) - (\omega_a \pm \omega_b \pm \omega_{IF})\,\tau_0(\beta_0,\gamma_0) + \omega_{LO}(\Delta x_1/c - \Delta y_1/c) \pm \omega_2(\Delta x_1/c + \Delta x_2/c - \Delta y_2/c) \pm \omega_{IF}(\tau_0 + \Delta x_1/c + \Delta x_2/c + \Delta z/c) \quad (3)$$

and $\omega_{\substack{u\\L}} = \omega_{LO} \pm \omega_2 \pm \omega_{IF}$

since $A_{\substack{u\\L}} = A_{\substack{u\\L}}^*$ and $B_{\substack{u\\L}} = B_{\substack{u\\L}}^*$.

Using the above equations, we may derive all properties of real interferometers except noise related ones. There are in fact so many attributes of these equations that we can cover only a few here.

We note that the R_r function above is real and still contains some easily varied instrumental parameters. Let us define a complex response function using the following considerations. Consider R_r at various values of θ, α

	θ	α	
(a)	$0°$	$0°$	$R_r = G_u T_u \cos\Phi_u + G_L T_L \cos\Phi_L$
(b)	90	0	$R_r = -G_u T_u \sin\Phi_u - G_L T_L \sin\Phi_L$
(c)	0	90	$R_r = -G_u T_u \sin\Phi_u + G_L T_L \sin\Phi_L$
(d)	90	90	$R_r = -G_u T_u \cos\Phi_u + G_L T_L \cos\Phi_L$

With this phase shifting sequence we may solve for each of the 4 parts of the complex functions

$$R_u = \int d\omega_{IF} \iint d\beta\, d\gamma \quad G_u(\omega_u)\, T_u(\beta,\gamma,\omega_u)\, e^{i\Phi_u}$$

$$R_L = \int d\omega_{IF} \iint d\beta\, d\gamma \quad G_L(\omega_L)\, T_L(\beta,\gamma,\omega_L)\, e^{i\Phi_L} \qquad (4)$$

For most of the rest of the discussion we will only consider one of these complex functions - say R_u. Let us also simplify equation (3) by taking normal values:

(a) $\tau(\beta,\gamma)$ can be a fairly rapid function of time so to "stop the fringes" we arrange for

$$\omega_a \pm \omega_b = \omega_{LO} \pm \omega_2$$

(b) To slow the phase changes even further and to keep Φ as independent of ω_{IF} as possible we insert a variable amount of delay (using pieces of cable + switching circuits) in the IF lines of one telescope (i.e. as Δz). Then, normally, we "track delay" meaning we keep

$$\Delta z = -\Delta x_1 - \Delta x_2 - c\,\tau_0$$

(c) Since ω_{LO} and ω_2 are independent of source (except for spectral line observations in which ω_2 is varied), we may lump the ω_{LO} and ω_2 terms into an "instrumental phase" which varies slowly with time (hopefully) as the Δx and Δy vary.

Setting $\omega = \omega_{LO} \pm \omega_{IF}$ we now have

$$R = e^{i\varphi} \int d\omega_{IF} \iint d\beta\, d\gamma\; G(\omega \pm \omega_{IF})\, T(\beta,\gamma,\omega \pm \omega_{IF})\, e^{i\Phi'} \qquad (5)$$

where

$$\Phi' = (\omega \pm \omega_{IF})(\tau - \tau_0)$$

Let us review the value of τ :

$$c\tau = \underline{B} \cdot \underline{S} = B_x \cos\delta \cos h + B_y \cos\delta \sin h + B_z \sin\delta$$

where $\underline{B}$ is the vector between one telescope and the other, $\underline{S}$ is the unit vector in the direction (β, γ), h is the source hour angle and δ the source declination. Thus

$$c(\tau - \tau_0) = \underline{B} \cdot \underline{S} - \underline{B_0} \cdot \underline{S_0}$$

1. Let us now "observe" a point source of known position ($\underline{S_0} = \underline{S}$) with a flat spectrum ($T$ independent ω) and use a narrow bandwidth. Then

$$T(\beta,\gamma) = T_0\, \delta(\beta,\gamma)$$

$$G(\omega \pm \omega_{IF}) = G_0\, \delta(\omega_{IF} - \omega_{IFO})$$

and

$$R = e^{i\varphi} G_0 T_0\, e^{i(\underline{B} - \underline{B_0}) \cdot \underline{S_0}(\omega + \omega_{IFO})/c}$$

If we observe several sources in rapid succession (e.g. φ stays ~ constant), we may then solve the set of equations

$$\text{observed phase} = \varphi + \Big[(B_x - B_{xo}) \cos\delta \cos h + (B_y - B_{yo}) \cos\delta \sin h + (B_z - B_{zo}) \sin\delta\Big](\omega \pm \omega_{IFO})/c$$

to determine the corrections to the assumed baseline. Since phase accuracies of ~ 1° can be obtained, one can measure $\underline{B}$ very accurately e.g.

$$\sigma(B) = \left(\frac{1^\circ}{57\ ^\circ/\text{radian}}\right)\frac{c}{2\pi f}$$

$$\sim \frac{1}{57}\ \frac{\lambda}{2\pi} = \frac{\lambda}{360}$$

where λ is the wavelength. Thus, if source positions are really accurately known, it is possible to measure distances to accuracies << 1 cm. NASA is currently engaged in projects which use interferometers in this way to try to measure small shifts in the earth's crust due to earthquakes and continental drift. Anyway, from now on we will assume that we can observe point sources to remove all effects of baseline error and of instrumental phase and will thus ignore those terms.

2. Let us now "observe" a point source at some position other than the assumed one - say $\beta = x_0$ $\gamma = y_0$. Keeping all other assumptions the same, the observed phase will be

$$\underline{B}\cdot(\underline{s}-\underline{s}_0) = B_x(\cos\delta\cos h - \cos\delta_0\cos h_0)$$
$$+ B_y(\cos\delta\sin h - \cos\delta_0\sin h_0) + B_z(\sin\delta - \sin\delta_0)$$

If we assume $\delta-\delta_0$ and $h-h_0$ are small, then to second order in these differences

$$\frac{\text{obs phase}}{\omega} = \Delta\delta\left[-B_x \sin\delta_0 \cos h_0 - B_y \sin\delta_0 \sin h_0 + B_z \cos\delta_0\right]$$
$$+ (\Delta h \cos\delta_0)\left[-B_x \sin h_0 + B_y \cos h_0\right]$$
$$+ \Delta\delta\,\Delta h\left[B_x \sin\delta_0 \sin h_0 - B_y \sin\delta_0 \cos h_0\right]$$
$$+ \frac{(\Delta\delta)^2}{2}\left[-B_x \cos\delta_0 \cos h_0 - B_y \cos\delta_0 \sin h_0 - B_z \sin\delta_0\right]$$
$$+ \frac{(\Delta h)^2}{2}\left[-B_x \cos\delta_0 \cos h_0 - B_y \cos\delta_0 \sin h_0\right]$$

This may be rewritten as

$$\frac{\text{obs phase}}{\omega} = ux + vy + wz$$

where
$$x = -\sin\Delta\alpha \cos(\delta_0 + \Delta\delta) \approx -\Delta\alpha \cos\delta_0 + \Delta\alpha\,\Delta\delta \sin\delta_0$$
$$y = \sin\Delta\delta + (1 - \cos\Delta\alpha)\sin\delta_0 \cos(\delta_0 + \Delta\delta)$$
$$\approx \Delta\delta + \tfrac{1}{2}(\Delta\alpha)^2 \sin\delta_0 \cos\delta_0$$
$$z = \cos\Delta\delta - 1 - (1 - \cos\Delta\alpha)\cos\delta_0 \cos(\delta_0 + \Delta\delta)$$
$$\approx -\tfrac{1}{2}\left[(\Delta\delta)^2 + (\Delta\alpha \cos\delta_0)^2\right]$$

and
$$u = -B_x \sin h_0 + B_y \cos h_0$$
$$v = -B_x \sin\delta_0 \cos h_0 - B_y \sin\delta_0 \sin h_0 + B_z \cos\delta_0$$
$$w = B_x \cos\delta_0 \cos h_0 + B_y \cos\delta_0 \sin h_0 + B_z \sin\delta_0$$

Note: This complicated second-order, three-dimensional formalism reduces in the first-order approximation to the simpler formulas derived in Part I of these notes. These complications are only important for large fields of view (i.e. large $\Delta\delta$, $\Delta\alpha$) observed with long baselines. As a result, the VLA must be concerned with these complications. However, we may deal with the simpler formulae:

$$x = \Delta h \cos\delta_0 \qquad = -(\alpha - \alpha_0)\cos\delta_0$$
$$y = \Delta\delta \qquad = (\delta - \delta_0)$$

Getting back to the point of this section — If we observe the point source at a number of hour angles, we may use the observed phases to derive values for $\Delta\alpha$ and $\Delta\delta$. The derivation of accurate source positions so that the radio sources might be identified optically was one of the original purposes of interferometers. Positions from the Cal Tech interferometer allowed optical astronomers to identify the first quasars.

3. Let's use the result of part 2 in another way. We've shown that Φ' of equation 5 (p 8) may be written as $(\omega + \omega_{IFO})(ux + vy)$. Thus, the interferometer responds to the Fourier transform of the brightness distribution, i.e.

$$R(u,v) = \iint_{\substack{\text{antenna}\\\text{beam}}} T(x,y)\, e^{2\pi i(\nu+\nu_{IFO})(ux+vy)/c}\, dx\, dy$$

Thus, we may reconstruct the brightness distribution by an inverse Fourier transform

$$T(x,y) = \iint_{-\infty}^{\infty} R(u,v)\, e^{-2\pi i(\nu+\nu_{IFO})(ux+vy)/c}\, \frac{du}{\lambda}\frac{dv}{\lambda}.$$

This technique for reconstructing the brightness distribution is known as "aperture synthesis". As explained in Part I it is complicated by the fact that we cannot measure $R(u, v)$ at all values of u and v.

I should point out that the $T(x, y)$ above is not the true brightness distribution but rather

$$T(x,y) = A(x,y)\, T_{true}(x,y)$$

where $A(x,y)$ is the single-dish antenna pattern.

4. So far we've considered interferometers of very narrow bandwidths only. Let's briefly relax that restriction. Equation 5 may be written, after calibration, as

$$R = \iint dx\, dy\; T(x,y)\, e^{i\omega(ux+vy)/c} \int d\omega_{IF}\; G(\omega_{IF})\, e^{i\omega_{IF}(ux+vy)/c}$$

when T is independent of ω. Let us assume $G(\omega_{IF})$ is centred on ω_{IF0}, rearrange some variables, and set

$$\frac{\lambda}{2\pi} = \frac{\omega + \omega_{IF0}}{c}$$

to obtain

$$R = \iint dx\, dy\; T(x,y)\, e^{2\pi i(ux+vy)/\lambda}\; g\big((ux+vy)/c\big)$$

where

$$g(z) = \int d\omega'_{IF}\; G(\omega'_{IF} + \omega_{IF0})\, e^{i\omega'_{IF} z}$$

is the Fourier transform of the bandpass function. Let us consider what happens for a point source at (x_0, y_0)

$$T(x,y) = T_0\; {}^2\delta(x - x_0, y - y_0)$$

or $$R = T_0\, e^{2\pi i(ux_0 + vy_0)(\omega+\omega_{IF0})/c}\; g\big((ux_0 + vy_0)/c\big) .$$

Let us now do the inverse Fourier transform:

$$T(x',y') = \int d\omega_{IF}\, G(\omega_{IF})\, T_0 \iint \frac{du\,dv}{\lambda^2}$$

$$\times \exp -\frac{2\pi i}{\lambda^2}\left[u\left(x' - \left(\frac{\omega+\omega_{IF}}{\omega}\right)x_0\right) + v\left(y' - \left(\frac{\omega+\omega_{IF}}{\omega}\right)y_0\right)\right]$$

$$= \int d\omega_{IF}\, G(\omega_{IF})\, T_0\ {}^2\delta\left(x' - \left(\frac{\omega+\omega_{IF}}{\omega}\right)x_0,\ y' - \left(\frac{\omega+\omega_{IF}}{\omega}\right)y_0\right)$$

Let $z = \left(\frac{\omega+\omega_{IF}}{\omega}\right) r_0$ where $r_0 = \sqrt{x_0^2 + y_0^2}$

and go to polar coordinates

$$T(r',\theta') = \frac{T_0\,\omega}{r_0}\int dz\, G'(z)\ {}^2\delta(r'\cos\theta' - z\cos\theta_0,\ r'\sin\theta' - z\sin\theta_0)$$

$$= \frac{T_0\,\omega}{r_0}\int dz\, G'(z)\ \delta(r'-z)\ \delta(\theta'-\theta_0)$$

$$= \frac{T_0\,\omega}{r_0}\ \delta(\theta-\theta_0)\ G\left(\omega_{IF} = \omega(r'-r_0)/r_0\right)$$

Thus the map is distorted by a function which is radial, which has width increasing with r_0, and amplitude decreasing with r_0. In general it may be shown that the finite bandwidth map is the infinitesimal map with each of its map points convolved with this function. This effect is important for the VLA where baselines are long and bandwidths rather wide.

5. No discussion of the consequences of our general equations would be complete without considering "double sideband" interferometers such as the continuum interferometer in Green Bank. For simplicity let us assume (see equations ② + ③ on page 6)

$$T_L = T_u \quad \text{— a true continuum source}$$

$$G_L = G_u \quad \text{— a good D.S.B. receiver}$$

$$G(\omega_{IF}) = G_0\,\delta(\omega_{IF} - \omega_{IF0}) \quad \text{— narrow bandwidth}$$

$$\left.\begin{array}{l} \omega_a = \omega_{LO} \\ \omega_b = \omega_2 = 0 \end{array}\right\} \text{true in Green Bank}$$

Then

$$R_r = G_0 \iint d\beta\, d\gamma\; T(\beta,\gamma) \left[\cos(\Phi_u + \theta) + \cos(\Phi_L + \theta)\right]$$

$$\Phi_{u \atop L} = (\omega_{LO} \pm \omega_{IF0})(\tau - \tau_0) + \omega_{LO}(\Delta x_1/c - \Delta y_1/c)$$
$$\pm\, \omega_{IF0}(\tau_0 + \Delta x_2/c + \Delta x_1/c + \Delta z/c)$$

or

$$R_r = G_0 \iint d\beta\, d\gamma\; T(\beta,\gamma) \cos a \cos b$$

$$a = \omega_{LO}(\tau - \tau_0) + \omega_{LO}(\Delta x_1/c - \Delta y_1/c) + \theta$$

$$b = \omega_{IF}(\tau_0 + \Delta x_2/c + \Delta x_1/c + \Delta z/c) + \omega_{IF0}(\tau - \tau_0)$$

Thus, the "fringes" represented by the a term are reduced in amplitude by $\cos b$. Normally $\omega_{IF0} \ll \omega_{LO}$. Thus, to prevent this reduction, we simply tracks delay (see page 7) —

$$\Delta z = -\Delta x_2 - \Delta x_1 - c\tau_0$$

and can ignore the $\omega_{IF0}(\tau - \tau_0)$ term. Thus, we're back to an equation for R_r just like the single sideband ones discussed earlier.