1

NRAO Summer Lecture
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PLASMA PHYSICS

References

(1) Jackson, *Classical Electrodynamics*, Ch. 10
(2) Tanenbaum, *Plasma Physics*, McGraw Hill
(3) Krall & Trivelpiece, *Principles of Plasma Physics*,
(4) Stix, *The Theory of Plasma Waves*

I. Introduction

Plasma Physics is the study of the dynamics of conducting fluids, usually taken to be ionized gases. The dynamics of such fluids are influenced by the presence of electric and magnetic fields, both externally imposed and internally generated. The interest of astrophysicists in plasma physics stems from the fact that most matter in the universe is ionized.

II. — Dynamical Formalism

A plasma is a many-body system, so mathematical descriptions must be based on some approximation scheme. The four most common formalisms are orbit theory, one fluid (magnetohydrodynamic (MHD) approach), two fluid, and kinetic theory or Vlasov approach. These represent a progression from least complete and elegant to the most so.

① Orbit Theory

Orbit Theory is essentially a single particle theory. It may be useful if external conditions rather than collective effects dominate the dynamics of the plasma. The equation of motion is:

$$m \frac{d\bar{v}}{dt} = m \overline{\nabla \Phi_G} + q\left(\bar{E} + \frac{\bar{v}}{c} \times \bar{B}\right) - \nu \bar{v} \qquad (1)$$

Φ_G = gravitational potential, ν = collision frequency

② One Fluid (MHD Theory)

The MHD approach considers a plasma to be a single, conducting fluid characterized by a density ρ, conductivity, σ, and velocity v. The equation of motion is:

$$\rho\left(\frac{\partial}{\partial t} + (\bar{v}\cdot\bar{\nabla})\right)\bar{v} = -\bar{\nabla} p + \frac{1}{c}\,\bar{J}\times\bar{B} + \mu \nabla^2 \bar{v} \qquad (2)$$

μ = viscosity

Equation (2) clearly needs additional equations for its solution, these are:

Continuity: $\frac{\partial \rho}{\partial t} + \bar{\nabla}\cdot(\rho\bar{v}) = 0$ (3)

Ohm's Law: $\bar{J} = \sigma\left(\bar{E} + \frac{1}{c}\bar{v}\times\bar{B}\right)$ (4)

Equation of State: $p = f(\rho)$ (5)

The MHD equations can describe a number

of "slow" plasma phenomena such as Alfvén waves and flows. The MHD approach cannot describe high frequency waves and charge separation phenomena.

3 Two Fluid Theory

A wider range of phenomena is accessible if one uses two rather than a single fluid. One fluid is the electrons, the other the ions. The equations may be set up in the form:

Continuity:
$$\frac{\partial n_\alpha}{\partial t} + \bar{\nabla} \cdot (n_\alpha \bar{v}_\alpha) = 0 \qquad (6)$$

Momentum:
$$n_\alpha m_\alpha \left(\frac{\partial}{\partial t} + \bar{v}_\alpha \cdot \bar{\nabla} \right) v_\alpha = n_\alpha q_\alpha \left(\bar{E} + \frac{1}{c} \bar{v}_\alpha \times \bar{B} \right) - \bar{\nabla} p_\alpha \qquad (7)$$

Equation of State: $p_\alpha = f(n_\alpha)$

Maxwell's Equations:

$$\bar{\nabla} \cdot \bar{E} = 4\pi \sum_\alpha n_\alpha q_\alpha \qquad (8)$$

$$\bar{\nabla} \times \bar{B} = \frac{1}{c} \frac{\partial \bar{E}}{\partial t} + \frac{4\pi}{c} \sum_\alpha n_\alpha q_\alpha v_\alpha \qquad (9)$$

4 Kinetic Theory or Vlasov Approach

The kinetic theory approach is the most elegant and useful, since it utilizes information on the velocity space distribution of the particles, wheras the two fluid equations posess information only on the moments of such distributions.

Given the phase space density $f_\alpha(x, y, z, v_x, v_y, v_z, t)$ of particles of species α, we have the Vlasov Equation:

$$\frac{\partial f_\alpha}{\partial t} + \bar{v}_\alpha \cdot \nabla f_\alpha + \frac{q_\alpha}{m_\alpha}\left(\bar{E} + \frac{1}{c}\bar{v}_\alpha \times \bar{B}\right) \cdot \nabla_v f_\alpha = 0 \quad (10)$$

and Maxwell's Equations.

$$\bar{\nabla} \cdot \bar{E} = 4\pi \sum_\alpha n_\alpha q_\alpha \int f_\alpha \, d^3 v_\alpha + 4\pi \rho_{ext} \quad (11)$$

$$\bar{\nabla} \times \bar{B} = \frac{1}{c}\frac{\partial \bar{E}}{\partial t} + \frac{4\pi}{c}\sum_\alpha \bar{n}_\alpha q_\alpha \int \bar{v} f_\alpha \, d^3 v + \frac{4\pi}{c} J_{ext} \quad (12)$$

III – Calculation of Basic Plasma Properties

We now use these formalisms to calculate some of the most fundamental properties of a plasma. The ones we will discuss are the plasma frequency, the gyrofrequency, the deBye length, the $\bar{E} \times \bar{B}$ drift, and the concept of adiabatic invariants.

(a) the plasma frequency

Use is made of the two-fluid equations, we assume an electrostatic, 0 temperature plasma. We then have:

$$\frac{\partial n_e}{\partial t} + \bar{\nabla}\cdot(n_e \bar{v}_e) = 0 \qquad \frac{\partial n_i}{\partial t} + \bar{\nabla}\cdot(n_i \bar{v}_i) = 0 \qquad (13)$$

$$n_e m_e \left(\frac{\partial}{\partial t} + \bar{v}_e\cdot\bar{\nabla}\right)\bar{v}_e = n_e q \bar{E}; \quad n_i m_i \left(\frac{\partial}{\partial t} + \bar{v}_i\cdot\bar{\nabla}_i\right)\bar{v}_i = -n_i q \bar{E} \qquad (14)$$

$$\bar{\nabla}\cdot\bar{E} = 4\pi q (n_e - n_i) \qquad (15)$$

We now apply conventional perturbation theory, $X = X_0 + \varepsilon X_1$, where ε is a smallness parameter, and retain only terms of first order in ε. The equations then become:

$$\dot{n}_{e1} + n_{e0}\,\bar{\nabla}\cdot\bar{v}_{e1} = 0 \qquad \dot{n}_{i1} + n_{i0}\,\bar{\nabla}\cdot\bar{v}_{i1} = 0 \qquad (13')$$

$$n_{e0} m_e \dot{\bar{v}}_{e1} = n_{e0} q \bar{E}_1 \qquad n_{i0} m_i \dot{\bar{v}}_{i1} = -n_{i0} q \bar{E}_1 \qquad (14')$$

$$\bar{\nabla}\cdot\bar{E}_1 = 4\pi q (n_{e1} - n_{i1}) \qquad (15')$$

Take divergence of equations (14') and we get:

$$n_{eo} m_e \frac{\partial}{\partial t} (\bar{\nabla} \cdot \bar{v}_{e1}) = n_{eo} q \, \bar{\nabla} \cdot \bar{E}_1$$

$$n_{io} m_i \frac{\partial}{\partial t} (\bar{\nabla} \cdot \bar{v}_{i1}) = - n_{io} q \, \bar{\nabla} \cdot \bar{E}_1 \qquad (16)$$

Substitute equation (15′) into (16) and we get:

$$-\ddot{n}_{e1} = \frac{4\pi q^2 n_{eo}}{m_e} (n_{e1} - n_{i1}); \quad \ddot{n}_{i1} = \frac{4\pi q^2 n_{io}}{m_i} (n_{e1} - n_{i1}) \qquad (17)$$

or

$$\ddot{n}_{e1} + \frac{4\pi q^2 n_{eo}}{m_e} (n_{e1} - n_{i1}) = 0$$

$$\ddot{n}_{i1} + \frac{4\pi q^2 n_{io}}{m_i} (n_{i1} - n_{eo}) = 0 \qquad (18)$$

These equations are of a kind commonly encountered in small oscillations. We introduce the normal co-ordinate $\xi \equiv (n_{e1} - n_{i1})$, and equation (18) becomes 1 equation:

$$\ddot{\xi} + \frac{4\pi q^2 n_0}{m_e} \left(1 + \frac{m_e}{m_i}\right) \xi = 0 \qquad (19)$$

This is the equation of an oscillation (not a wave), with a frequency,

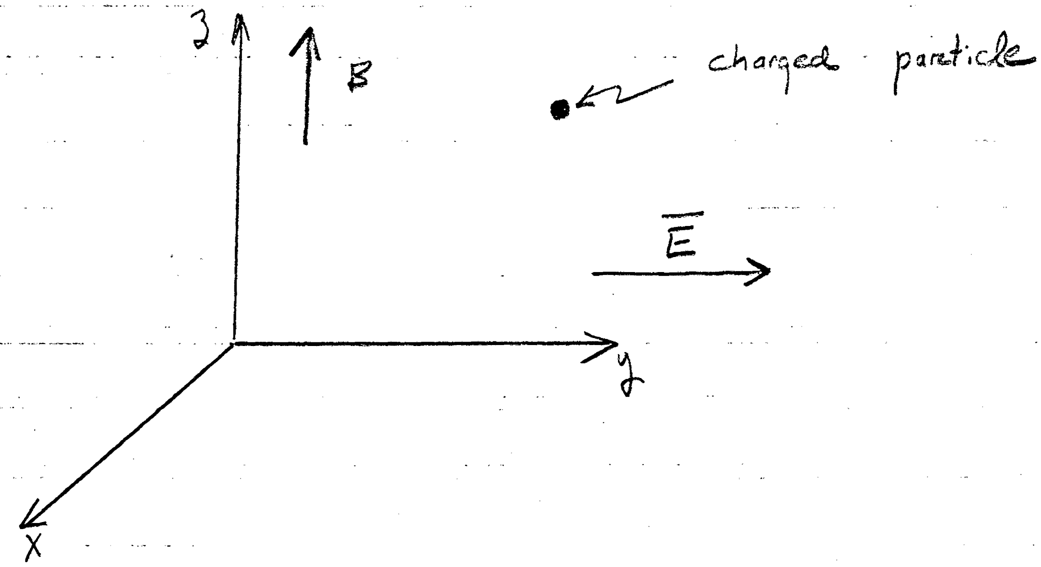
$$\omega_p^2 = \frac{4\pi q^2 n_0}{m_e} \left(1 + \frac{m_e}{m_i}\right) \qquad (20)$$

This is the basic eigen frequency of an unmagnetized cold plasma.

(b) Gyrofrequency and $\bar{E}\times\bar{B}$ Drifts

Use is made of orbit theory.

Consider a particle subject to a magnetic field in the z direction and an electric field E in the y direction, as shown below.



If we ignore collisional and gravitational forces, equation (1) gives us:

$$m\dot{\bar{v}} = q\left(\bar{E} + \frac{1}{c}(\bar{v}\times\bar{B})\right) \qquad (21)$$

Breaking this into components, we get:

$$\left.\begin{aligned} m\dot{v}_z &= 0 && (a)\\ m\dot{v}_y &= q\left(E_o - \frac{1}{c}v_x B_o\right) && (b)\\ m\dot{v}_x &= \frac{q}{c}v_y B_o && (c)\end{aligned}\right\} \qquad (22)$$

Differentiate 22-b with respect to time and substitute 22-c, we obtain:

$$\ddot{v}_y + \left[\frac{qB_0}{mc}\right]^2 v_y = 0 \qquad (23)$$

Thus the motion in the y direction is simple harmonic motion with frequency

$$\omega_G = \frac{qB_0}{mc} \quad \text{(the gyrofrequency)} \qquad (24)$$

Doing likewise with 22-c, we obtain:

$$\ddot{v}_x = \frac{q}{m}\omega_G\left(E_0 - \frac{1}{c}v_x B\right) \qquad (25)$$

$$\ddot{v}_x + \omega_G^2 v_x = \left(\frac{q}{m}\right)\omega_G E_0 \qquad (25')$$

The homogeneous solution is just that of equation (23). The simplest inhomogeneous solution is $v_x = c\,(E_0/B_0)$. Thus the motion in the x direction is the superposition of harmonic motion and a drift velocity.

(c) The De Bye Length

If a charge is introduced in a plasma, the electrons and protons will move to shield it. We may investigate this using the two fluid equations. Using equations 6-9, ignoring magnetic effects, retaining pressure terms, and once again linearizing the equations, we obtain:

$$n_{eo} m_e \dot{\bar{v}}_{e1} = -\bar{\nabla} p_e + n_{eo} q \bar{E}_1 \; ; \quad n_{io} m_i \dot{\bar{v}}_i = -\bar{\nabla} p_i - n_{io} q \bar{E}_1 \quad (26)$$

$$\bar{\nabla}\cdot\bar{E}_1 = 4\pi q (n_{e1} - n_{i1}) \qquad (27)$$

After transients have passed; $\dot{\bar{v}}_{e1} = \dot{\bar{v}}_{i1} = 0$

$$\therefore \quad \frac{1}{n_{eo} m_e} \bar{\nabla} p_e = \frac{q}{m_e} \bar{E}_1 \; ; \quad \frac{1}{n_{io} m_i} \bar{\nabla} p_i = -\frac{q}{m_i} \bar{E}_1 \qquad (28)$$

assume $T_e = T_i = T$, then:

$$\bar{\nabla} p_e = KT \, \bar{\nabla} n_e \, , \quad \bar{\nabla} p_i = KT \, \bar{\nabla} n_i \qquad (29)$$

then; $$\bar{\nabla} n_{e1} = \frac{q n_{eo}}{KT} \bar{E}_1 \; ; \quad \bar{\nabla} n_i = -\frac{q n_{io}}{KT} \bar{E}_1 \qquad (30)$$

Take the divergence of equations (30), and we get:

$$\nabla^2 n_{e1} = \frac{q n_{eo}}{KT}(\bar{\nabla}\cdot\bar{E}_1) \; ; \quad \nabla^2 n_{i1} = -\frac{q n_{io}}{KT}(\bar{\nabla}\cdot\bar{E}_1) \qquad (31)$$

substituting eq. (27)

$$\nabla^2 n_{e1} = \frac{4\pi q^2 n_{eo}}{KT}(n_{e1} - n_{i1}) \; ; \quad \nabla^2 n_{i1} = -\frac{4\pi q^2 n_{io}}{KT}(n_{e1} - n_{i1}) \qquad (32$$

defining $\xi = n_{e1} - n_{i1}$, equation (32) becomes

$$\nabla^2 \xi = \frac{8\pi q^2 n_{eo}}{KT}\,\xi \qquad (33)$$

The solution to this is:

$$\left.\begin{aligned} \xi &= \frac{A}{r} e^{-r/\lambda} \\ \text{where} \quad \lambda &= \sqrt{\frac{KT}{8\pi q^2 n_{eo}}} \end{aligned}\right\} \qquad (34)$$

Thus a plasma will shield a test charge over a distance of about one DeBye length.