## THE RELATIONSHIP BETWEEN ANTENNA SITES ON THE ARMS OF THE WYE

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## INTRODUCTION

This memorandum discusses two methods fer alignment of the telescopes at the various sites along the arms of the wye.

1. Align the vertical axis of the telescope to local gravity and the azimuth zero to north.
2. Align the vertical axis of the telescope parallel to gravity at the wye center and the azimuth zero parallel to the north at the wye center.

## Local Coordinates of Each Antenna Site:

If each antenna site is aligned to local gravity and north then the zenith angle and hour angle is different for each antenna site. Refer to Figure 1. Let $N$ be the north pole of the earth sphere. $90-\phi_{1}$ is a sector of a great circle which passes through the north pole and the center of the wye. $90-\phi_{2}$ is a sector of a great circle which passes through the north pole and a telescope site on the southeast arm of the wye. $C^{\prime}$ is a sector of $a$ great circle which passes through the wye configuration center and the antenna site. $b$ is a sector of a great circle which passes through the antenna and is at right angles to the great circle containing the sector $90-\phi_{1}$.

The great circle containing sector $90-\phi_{1}$ and $90-\phi_{2}$ are the meridians of the wye center and the telescope site respectively. The dotted lines represent the latitude of the wye center and the telescope site. $\beta^{\prime}$ is the spherical angle the great circle containing the southeast arm makes with the meridian passing through
the wye center measured from south as shown. $\beta$ is the spherical angle between the two meridians.

From spherical trigonometry and reference to Figure 1:

$$
\begin{equation*}
\sin b=\sin \beta^{\prime} \sin C^{\prime} \tag{1}
\end{equation*}
$$

(Angles $b$ and $C^{\prime}$ are small $\left(b, C^{\prime}<12 \mathrm{~min}\right.$ ). We can substitute $b=\sin b$ and $C^{\prime}=\sin C^{\prime}$ with negligible error. For example: If $\xi \leq 12$ min $|\sin \xi-\xi| \leq \varepsilon$ where $\varepsilon \leq 1 \times 10^{-8}$ )
$b=C^{\prime} \sin B^{\prime}$ may be written from eq. l with negligible error (2)

Again from spherical trigonometry we may write

$$
\begin{equation*}
\tan B=\frac{\tan b}{\sin \left(\ell+90-\phi_{1}\right.} \tag{3}
\end{equation*}
$$

As before, we may substitute $\tan \beta=\beta$ and tan $b=b$ with negligible error. Making the substitution into equation 3 we may write

$$
\begin{equation*}
B=\frac{b}{\sin \left(\ell+90-\phi_{1}\right)} \tag{4}
\end{equation*}
$$

Upon substituting equation 2 into equation 4 and letting $\ell=\sin \ell$
and $\cos \ell=1\left(|\cos \ell|-1 \leq 1 \times 10^{-5}\right.$ if $\ell \leq 14$ min $)$ and expanding sin $\left(\ell+90-\phi_{1}\right)$. we can write

$$
\begin{equation*}
\beta=-\frac{C^{\prime} \sin \beta^{\prime}}{\ell \cos \left(90-\phi_{1}\right)+\sin \left(90-\phi_{1}\right)} \tag{5}
\end{equation*}
$$

From figure 1

$$
\begin{equation*}
\ell=C^{\prime} \cos \beta^{\prime} \tag{6}
\end{equation*}
$$

Substituting equation 6 into equation 5 and making use of the identities $\cos \left(90-\phi_{1}\right)=\sin \phi_{1}$ and $\sin \left(90-\phi_{1}\right)=\cos \phi_{1}$,

$$
\begin{equation*}
B=\frac{C^{\prime} \sin \beta^{\prime}}{C^{\prime} \cos \beta^{\prime} \sin \phi_{1}+\cos \phi_{1}} \tag{7}
\end{equation*}
$$

Equation 7 gives $\beta$ in terms of the known parameters $C^{\prime}, \beta^{\prime}$, and Knowing $\beta$ we can solve for ( $90-\phi_{2}$ ) using the law of sines for spherical trigonometry

$$
\begin{equation*}
\frac{\sin \left(180-\beta^{\prime}\right)}{\sin \left(90-\phi_{2}\right)}=\frac{\sin \beta}{\sin C^{\prime}} \tag{8}
\end{equation*}
$$

Upon rearranging equation 8

$$
\begin{equation*}
\cos \phi_{2}=\frac{\sin C^{\prime} \sin \beta^{\prime}}{\sin \beta} \tag{9}
\end{equation*}
$$

Substituting $C^{\prime}=\sin C^{\prime}$ and $B=\sin \beta$

$$
\begin{equation*}
\phi_{2}=\cos ^{-1}\left[\frac{C^{\prime} \sin B^{\prime}}{\beta}\right] \tag{10}
\end{equation*}
$$

Example:
Locate the center of the wye at $108^{\circ} 00^{\prime} 00^{\prime \prime}$ west longitude and $34^{\circ} 00^{\prime} 00^{\prime \prime}$ north latitude. The latitude and longitude of a telescope site $21,000.00$ meters from the wye center on the southeast arm may be calculated from equations 7 and 10 .
(Equation 7):

$$
\beta=\frac{C^{\prime} \sin \beta^{\prime}}{C^{\prime} \cos \beta^{\prime} \sin \phi_{1}+\cos \phi_{1}}
$$

(Equation 10):

$$
\phi_{2}=\operatorname{Arccos}\left(\frac{C^{\prime} \sin \beta^{\prime}}{\beta}\right)
$$

Given:

$$
\begin{aligned}
& B^{\prime}=65^{\circ} 00^{\prime} 00^{\prime \prime} \\
& C^{\prime}=d / R
\end{aligned}
$$

Where $d$ is the distance from the center of the wye to the telescope site and $R$ is the radius of the earth calculated below:

Earth mean radius at the poles $6,356,912$ meters
Earth mean radius at the equator $6,378,338$ meters
Consider the cross section of the earth taken through the site meridians as an ellipse and calculating the radius at $34^{\circ}$ latitude and adding 7,000 feet for site elevation

$$
\begin{aligned}
\mathrm{R} & =6,373,802 \text { meters } \\
C^{\prime} & =00^{\circ} 11^{\prime} 20^{\prime \prime} \\
\phi_{1} & =34^{\circ} 00^{\prime} 00^{\prime \prime} \\
\operatorname{Cos} \beta^{\prime} & =0.42261826 \\
\operatorname{Sin} B^{\prime} & =0.90630779 \\
\operatorname{Sin} \phi_{1} & =0.55919290 \\
\operatorname{Cos} \phi_{1} & =0.82903757
\end{aligned}
$$

From equations 7 and 10 :

$$
\begin{aligned}
& B=00^{\circ} 12^{\prime} 22^{\prime \prime} \\
& \phi_{2}=33^{\circ} 55^{\prime} 12^{\prime \prime}
\end{aligned}
$$

The telescope sight on the southeast arm of wye 21,000 meters from the center will have location coordinates of $107^{\circ} 47^{\prime} 44^{\prime \prime}$ west longitude and $33^{\circ} 55^{\prime} 12^{\prime \prime}$ north latitude which compares to $108^{\circ} 00^{\prime} 00^{\prime \prime}$ west longitude and $34^{\circ} 00^{\prime} 00^{\prime \prime}$ north latitude for the wye center.

## Telescope Coordinate Transformation:

The telescope antennas currently under consideration for the VLA system have a coordinate system of elevation over azimuth. The source position is known in hour angle and declination. A coordinate transformation is therefore required to convert hour angle and declination to the telescope coordinates.

Refer to Figure 2:
a. $\mathbf{P}_{1}$ is the north celestial pole
b. $\mathrm{P}_{2}$ is the south celestial pole
c. Horizontal plane is a plane tangent to the earth sphere at the location of a telescope projected to the center of the earth sphere.
d. $Z$ is the zenith angle from the local zenith to a source.
e. $H$ is the local hour angle of a source referred to the local meridian.
f. $\phi$ is the latitude of the local telescope site.
g. $\delta$ is the declination of a source.
h. a is the local azimuth angle of a source referenced to local south.
i. $h$ is the local elevation angle of a source referenced to the local tangent plane.
If the position of a source is known in terms of local hour angle and declination the following coordination transformation equations will give the source position in terms of telescope azimuth and zenith angle a and 2 .

$$
\begin{align*}
& Z=\operatorname{Arc} \cos (\sin \phi \sin \delta+\cos \phi \cos \delta \cos H)  \tag{11}\\
& a=\operatorname{Arc} \sin (\cos \delta \sin H / \sin Z) \tag{12}
\end{align*}
$$

The derivative of azimuth and elevation angle with respect to time is given by the following equations.

$$
\begin{align*}
& \frac{d Z}{d t}=\cos \phi \sin a \frac{d H}{d t} \\
& d H / d t=15 / 3600 \text { deg/sec } \quad \text { (Sidereal time) } \\
& \frac{d Z}{d t}=\cos \phi \sin a 15 / 3600 \mathrm{deg} / \mathrm{sec}  \tag{13}\\
& d a / d t=[\sin \phi+\cos \phi \cot Z \cos a] 15 / 3600 \mathrm{deg} / \mathrm{sec}  \tag{14}\\
& d^{2} a / d t^{2}=-\cos \phi\left[\frac{\cos a}{\sin ^{2} Z} \frac{d Z}{d t}+\cot Z \sin a \frac{d a}{d t}\right]
\end{align*}
$$

Figures 3 through 7 are plots of equations 12,14 , and 15 for different source declinations ( $\delta$ ) as a function of hour angle (H).

The hour angle of a source in respect to the center of the wye will differ from that of telescope site on an arm of the wye by amount related to the difference between the respective longitude and latitude.

## Telescope Position Commands

It is evident from the foregoing discussion that the azimuth and zenith angles required to track a source are different for each telescope site, dependent upon the local latitude and longitude of the site. However, if the azimuth and elevation axes of each telescope were aligned parallel to a reference at the center of the wye it would be possible to operate all telescopes of the VLA system with one coordinate transformation calculation.

It is of interest to calculate the pointing error which would result should each telescope be aligned relative to local gravity and north and be given a pointing command for the center of the wye.

Refer to Figure 8:

```
\(\phi_{1}=\) the latitude of the wye center
    \(\phi_{2}=\) the latitude of the telescope site
    \(\mathrm{H}_{1}=\) the hour angle of the wye center (source)
    \(\mathrm{H}_{2}=\) the hour angie of the telescope site (calculated)
    \(\mathrm{Za}=\) the commanded zenith angle
    \(\mathrm{A}=\) the commanded azimuth angle
    \(\delta_{1}=\) the source declination
    \(\delta_{2}=\) the declination of the telescope beam center
    \(\xi=\) the hour angle error
    \(\Delta \delta=\) the declination error
    \(\Delta \mathrm{H}=\) the hour angle between the wye center meridian and
        the telescope site meridian
        \(\xi=\left(\mathrm{H}_{1}+\Delta \mathrm{H}\right)-\mathrm{H}_{2}\)
        \(\Delta \delta=\delta_{1}-\delta_{2}\)
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Equations 16 and 17 are the errors in pointing direction in terms of hour angle and declination.
$\mathrm{H}_{1}$ is the independent variable
$\Delta H=\beta$ may be calculated by equation 7 ctpendent upon the location of the telescope relative to the wye center.
$\delta_{1}$ is given for a source
Upon substituting $A=180-a$ in equation 12, we get:

$$
\begin{equation*}
A=\operatorname{Arc} \sin \left(\cos \delta_{1} \sin H_{1} / \sin Z\right) \tag{18}
\end{equation*}
$$

From the law of cosines -

$$
\begin{equation*}
\sin \delta_{2}=\cos Z \sin \phi_{2}+\sin Z \cos \phi_{2} \cos A \tag{19}
\end{equation*}
$$

Where Z may be obtained from equation 11 .
Upon rearranging equation 18 and substituting $H_{2}$ for $H_{1}$ and $\delta_{2}$ for $\delta_{1}$ :

$$
\begin{equation*}
H_{2}=\operatorname{Arcsin}\left(\sin A \sin Z / \cos \delta_{2}\right) \tag{20}
\end{equation*}
$$

From equation 19:

$$
\begin{equation*}
\delta_{2}=\arcsin \left(\cos Z \sin \phi_{2}+\sin Z \cos \phi_{2} \cos A\right) \tag{21}
\end{equation*}
$$

The total angular error " $P$ " using the law of cosine and referring to figure 8:

$$
\begin{align*}
& \cos P=\cos \left(90-\delta_{1}\right) \cos \left(90-\delta_{2}\right)+\sin \left(90-\delta_{1}\right) \sin \left(90-\delta_{2}\right) \cos \xi \\
& \cos P=\sin \delta_{1} \sin \delta_{2}+\cos \delta_{1} \cos \delta_{2} \cos \xi \tag{22}
\end{align*}
$$

Where

$$
\xi=\mathrm{H}_{1}+\Delta \mathrm{H}-\mathrm{H}_{2}
$$

and
$\Delta H=\beta$ which may be calculated from equation 4
Figure 9 is a plot of $P$ vs hour angle ( $H_{1}$ ) for the wye location of the example on page 3.

CONCLUSIONS
It is obvious that each telescope should not he aligned to local gravity and north and then be given an azimuth and zenith angle command calculated for the local coordinates at the wye center. (Refer to fig. 9 which shows the error that would result.) Therefore if alignment is made to the local gravity and north the coordinate calculations eq. 11 and 12 will be required for each source position updating for each telescope.

If the source position is updated 10 times each second and there are 36 telescopes, 72 calculations will be required each second. If the telescope servo system requires rate information, equations 13 and 14 , the number of calculations required will double to 144 per second.

Perhaps another and more important problem with local telescope alignment is that the computer program must be changed each time a telescope position is changed. If the change is overlooked, errors will result, given by equatinns 16, 17, and 22. These errors are quite large (See fig. 9) and could cause the telescope beam to be completely off the source.

Aligning each telescope site to a we center reference will reduce the coordiante transformation calculations to 2 (or 4 if rate information is required). The problem of establishing the proper alignment should be no more difficult than aligning to the local gravity and north.

Unless there are other telescope pointing problems (i.e. correction curves), that make a local gravity and north coordinates system alignment desirable, it would seem that a coordinated system alignment to the wye center is preferred.





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FIG. 8 SPHERICAL TRIANIES PROVENTED ON THE CELESTRIA SPHERE. SHOWING A RADIO SOURCE *, WYE ZENITH O, ANO TELESCOPE ZENITH ©


FIG. 9 FROE BETWEEN THE POSITION OF A SOURCE 21 RM FROM WYE CENTER ON THE SOUTHEAST ARM:.

