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AN ANALYSIS OF SIGNAL PROCESSING IN THE VLA CORRELATOR

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It is the task of the correlator subsystem of the VLA to accept analog IF signals at $\lesssim$50 MHz bandwidth and to produce estimates of the complex cross-correlation coefficients for various pairs of such signals; in spectrometer modes, the correlator must produce the correlation coefficients separately for each of K slightly-overlapping sub-bands of the input bandwidth where K is adjustable.

The implementation which has been chosen is almost entirely digital, and is based on three-level quantization of the IF signals. In this memo, the requirements for such an implementation are analyzed with the intention of providing guidelines for design. Some of the design details which have been discussed in other reports (e.g. VLA Electronics Memo No. 131 by A. M. Shalloway) are considered here, but in general this report attempts to avoid specifying particular methods of implementing the required logic.

1. Three-Level Multipliers and Accumulators

In an earlier memo (D'Addario 1975), it was shown that the standard deviation of the count obtained from a three-level multiplier/accumulator operating optimally on Nyquist-sampled, uncorrelated, white Gaussian noise signals is:

$$\sigma_d = .5407 \sqrt{V_s} \tag{1}$$

where V_s is the number of pairs of samples multiplied. If the signals are partially correlated, non-white, or oversampled, then σ_d is larger.

In the VLA correlator, it is planned to accumulate the V_s samples in two stages: R samples will be accumulated in a counter, after which the counter's contents will be added to a word in a larger memory; this will be done V_s/R times. At each dump of the counter, it will be rounded at the (k+1)st bit, so that the least significant k bits are discarded. The standard deviation in the final count due to the roundoff errors thus introduced is:

$$\sigma_r = \frac{2^k}{\sqrt{12}} \sqrt{V_s/R} \tag{2}$$

If we require $\sigma_r/\sigma_d \leq 0.1$, then we find

$$k \leq \log_2 (.05407\sqrt{12R})$$

$$\boxed{k \leq \tfrac{1}{2} \log_2 R - 2.42} \tag{3}$$

The word length required in the accumulating memory is determined as follows. From probability tables (National Bureau of Standards 1959), we find that the expected count for 100% correlation is 1.5485 V_s, for 50% correlation it is 1.2260 V_s (assuming optimum thresholds), and for zero correlation it is 1.0 V_s. If we want to be able to measure correlation coefficients approaching 1.0, the number of bits required is:

$$b = \text{gif } (\log_2 0.5485\, V_s) + 1 \qquad (\rho_{max} = 1) \tag{4}$$

where gif(x) is the largest integer not exceeding x. If the maximum correlation coefficient to be measured is 0.5, the number of bits required reduces to

$$b = \mathrm{gif}\ (\log_2 0.2260\ V_s) + 1 \qquad (\rho_{max} = 0.5). \tag{5}$$

Note that it is not necessarily required to have enough bits to represent V_s, much less $2V_s$. Although overflow can occur, no information is lost. The reader may convince himself of this by considering a few examples.

We also have

$$V_s = 2TW, \tag{6}$$

where T is the integrating time and W is the Nyquist bandwidth (half the sampling rate).

Correlation coefficients greater than 0.5 are expected only for the Sun and perhaps for some pulsars (flux density > 200 Jy at 21 cm). In these cases, shorter than maximum integrating times can generally be used. Thus it may be scientifically satisfactory to restrict the maximum observable correlation to 0.5 at maximum bandwidth and integrating time; larger correlation coefficients could still be measured by reducing the bandwidth and/or the integrating time. However, consultation with potential observers should be undertaken before such a decision is made.

If we suppose that the worst-case mode has W = 3.125 MHz, T = 38.77 s, and $\rho_{max} = 0.5$, then from (5)

$$b = 26\ .$$

Taking R = 8192, (3) gives

$$k \leq 4.$$

The word length in the memory is then $\boxed{b - k = 22 \text{ bits}}$. The effect of the use of this word length on the available bandwidths and integrating times is illustrated in Figure 1.

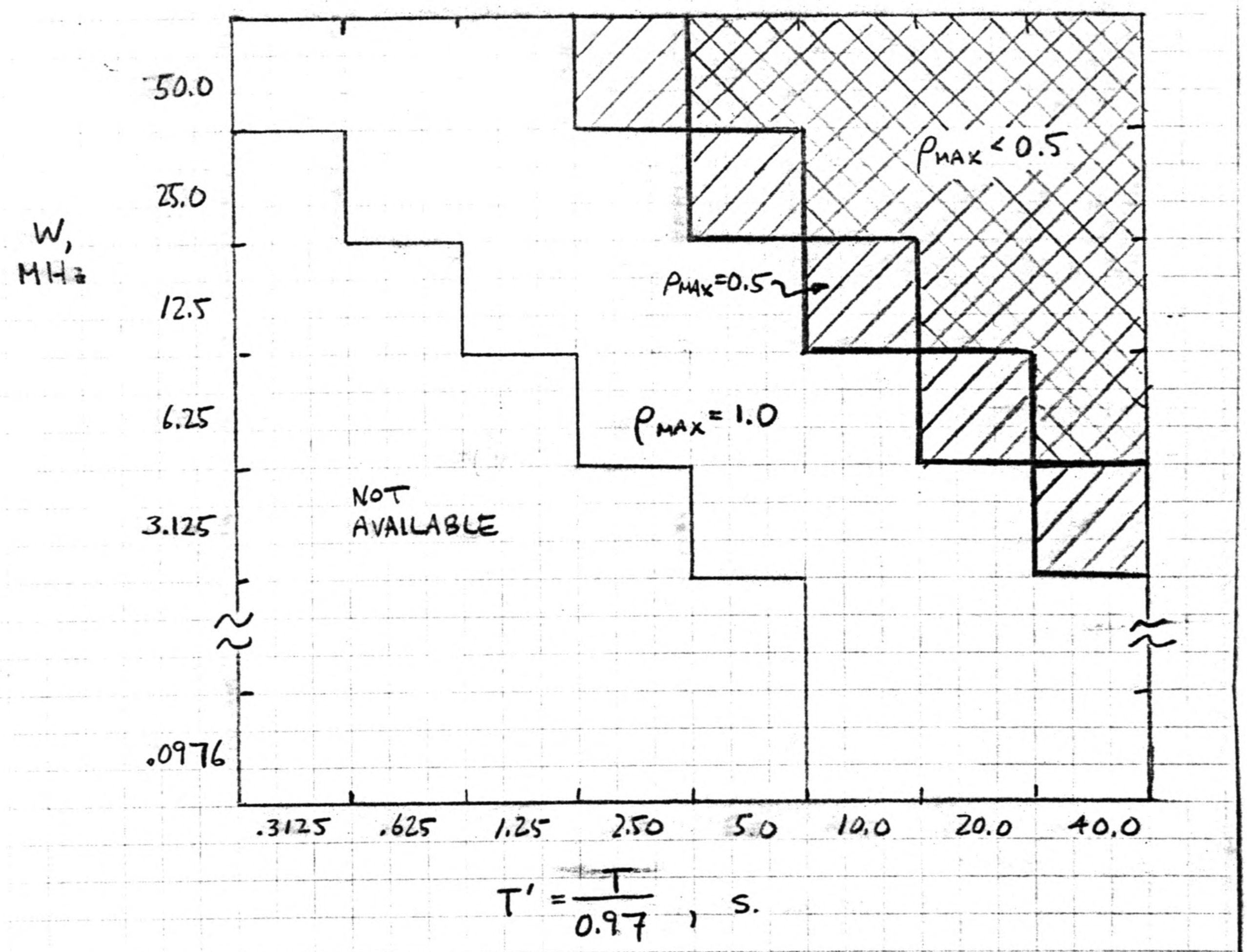

Figure 1

For 22-bit memory and 4 bits discarded at counter (R = 8192), chart showing maximum measureable correlation coefficient before overflow as a function of bandwidth and integrating time. (T' is the real time per integrating period at 97% efficiency.)

2. Standardization of Counts for Various Modes of Operation

The average count obtained in any correlator depends on the correlation coefficient of the signals and on the quantization thresholds, but is directly proportional to V_s. Since the correlator is designed to operate over a wide range of values of W and T, the scale of the count relative to the correlation coefficient being measured will be variable. It is desirable that the count be normalized to a standard scale very early in the processing, so that the later operations can be implemented without regard to scale. Whereas the various available values of T and W all differ by factors which are powers of two, the required scaling can be accomplished by bit shifting. Letting C_{ij} and C'_{ij} be the count for a particular correlator before and after scaling, we have

$$C'_{ij} = 2^{\ell} C_{ij} = \frac{V_o}{V_s} C_{ij}, \quad \ell \in \{0, \pm 1, \pm 2,\} \ldots \quad , \tag{7}$$

where V_o is the largest number of samples for which a correlation coefficient of 1.0 is to be measurable. [For example, with the 22-bit memory of the last section, $V_o = 2\ (1.5625\ \text{MHz})\ (38.77\ \text{s}) = 121{,}156{,}250 \approx 2^{26} + 2^{25} + 2^{24}$]

For small correlation coefficients, C'_{ij} will be near V_o. In order to minimize the word length required to maintain a given precision in later processing, it is desirable to further standardize by replacing C'_{ij} with

$$C''_{ij} = C'_{ij} - V_o . \tag{8}$$

The subtraction should probably be done in two's complement arithmetic for compatibility with most minicomputers. Note that $\left\langle C''_{ij} \right\rangle = 0$ for zero correlation coefficient.

Bits which are shifted off the MSB end of the word in computing C'_{ij} may be ignored; similarly, high order bits of V_o may be ignored. Notice that if V_s is a power of 2, then V_o can be chosen so that C''_{ij} is obtained from C'_{ij} by merely complementing its MSB (assuming two's complement arithmetic).

(In our example with the 22-bit memory, V_s is not a power of two. But suppose that the efficiency is slightly reduced so that $V_o = 2^{26} + 2^{25} + 2^{2\cdot}$ $= 117{,}440{,}512$. Then, recalling that $b = 26$ is the total number of accumulator bits so that the MSB of the memory word represents 2^{25} counts, we see that the two's complement representation of $-V_o$ is $0100\ldots0_2$. Thus, subtraction of V_o affects only the two MSB's of the word.)

After standardization, additional rounding may be done to reduce the word length without significant loss of information. Suppose that the b bits of the total accumulator (of which the least significant k bits were discarded at each dump) are finally rounded at the $(\ell + 1)$st bit, so that now ℓ bits altogether have been discarded. The standard deviation of this roundoff error is then

$$\sigma_R = \frac{2^\ell}{\sqrt{12}} \tag{9}$$

and if we again require $\sigma_R \le 0.1\ \sigma d$, we find

$$\boxed{\ell \le \tfrac{1}{2} \log_2 V_s - 2.42 .} \tag{10}$$

Thus finally we retain $b^* = b - \ell$ bits. The standardization will make these bits the most significant ones in the word. In the worst case we can take $V_s = V_o$ (cf. definition of V_o above), whence

$$\begin{aligned} b^* &= b - \ell \\ &= \text{gif}\,(\log_2 .5485\, V_o) + 1 \quad - (\tfrac{1}{2} \log_2 V_o - 2.42) \end{aligned} \tag{11}$$

[In our example with $V_o = 2^{26} + 2^{25} + 2^{24}$, we find $b^* = 15$ bits.]

3. Quantization Corrections

The standardized count C''_{ij} can now be related to the cross-correlation coefficient ρ_{ij} of the corresponding two signals. The relationship has been studied by Cooper (1970) and further by Thompson (1973). When $\rho_{ij} \ll 1$, the two are very nearly proportional, with the proportionality constant depending non-linearly on the quantization thresholds. (Assumetry in the two thresholds of a sampler causes a d.c. offset, but in the VLA this is eliminated by phase switching and we shall not consider it further.) Thompson (1973) showed that the threshold dependence can be eliminated to first order in the threshold errors by computing

$$R_{ij} = \frac{2\, C''_{ij}}{C''_{ii} + C''_{jj}} \tag{12}$$

where C''_{ii}, C''_{jj} are standardized counts of self-correlators operating on the signals being cross-correlated. R_{ij} then remains proportional to ρ_{ij} to within 1% of ρ_{ij} for threshold errors up to 10%.

There are two problems with this approach. First, the variation with threshold is < 1% only for small correlation coefficients, $\rho_{ij} \lesssim 0.3$ (correlated flux $\lesssim$ 180 Jy), and it will probably be important to be able to observe stronger sources with high accuracy. This is demonstrated in Table I, which was constructed from probability tables (NBX, 1959). Secondly, it is highly desirable to work to accuracies much better than 1%. The latter is expected to be the ultimate accuracy of the maps, limited primarily by atmospheric and local oscillator phase variations; contributions from other sources should be kept much smaller.

These considerations suggest that the non-linearity of the ρ_{ij} <u>vs</u>. C''_{ij} curve at fixed thresholds and the dependence of C''_{ij} on the thresholds at fixed ρ_{ij} should be considered simultaneously, and furthermore that a higher-than-first-order approximation must be used if threshold errors up to 10% are allowed. Thus we have the function

$$\rho_{ij} = \rho_{ij}\,(C''_{ij},\, C''_{ii},\, C''_{jj}),$$

which can be calculated for any values of its arguments by a long numerical procedure or by reference to tables. Table lookup may prove to be the best

TABLE I-a. STANDARDIZED COUNTS FOR VARIOUS THRESHOLD SETTINGS

	$V_i=0.5$, $V_j=0.5$	$V_i=0.5$, $V_j=0.6$	$V_i=0.5$, $V_j=0.7$	$V_i=0.6$, $V_j=0.6$	$V_i=0.7$, $V_j=0.7$	} THRESHOLD / σ (NOMINAL = 0.612)
$\rho_{ij}=0.1$	.0498	.0468	.0440	.0444	.0390	
0.2	.0996	.0942	.0682	.0890	.0782	VALUES OF
0.5	.2540	.2396	.2234	.2260	.1972	C_{ij}''/V_0
0.9	.4890			.4286	.3178	
1.0	.6171	.5485	.4839	.5485	.4839	
C_{ii}''/V_0 :	.6171	.6171	.6171	.5485	.4839	
C_{jj}''/V_0 :	.6171	.5485	.4839	.5485	.4839	

TABLE I-b. RATIOS OF CROSS- TO SELF-CORRELATOR COUNTS

VALUES OF $R_{ij} = 2C_{ij}''/(C_{ii}'' + C_{jj}'')$

$\rho_{ij}=0.1$	.0807	.0803	.0799	.0809	.0806	(± 0.5%)
0.2	.1614	.1616	.1602	.1622	.1616	(± 0.6%)
0.5	.3970	.4111	.4058	.4120	.4075	(± 2%)
0.9	.7924			.7814	.7683	
1.0	1.0000	.9411	.8790	1.0000	1.0000	(± 13%)

practical solution, but we should also consider whether a simple formula can approximate the function satisfactorily. I suspect that 0.1% accuracy can be obtained with a polynomial of the form.

$$\hat{\rho}_{ij} = (\alpha\, C''_{ij} + \beta\, C''^{\,3}_{ij})\ [1 + \gamma\, (C''_{ii} + C''_{jj}) + \delta\, (C''^{\,2}_{ii} + C''^{\,2}_{jj})]$$

where α, β, γ, and δ are predetermined constants, but I have not yet tried it.

Regardless of how it is done, an estimate of ρ_{ij} must be obtained from the correlator counters, free of non-linearities induced by the quantization. This must be done for each correlator separately before they are combined in any way, in particular before they are Fourier transformed from the lag to the frequency domain for spectroscopy.

Whether the computation can be done in a minicomputer or must be done in special purpose hardware depends on the available computing time. In the worst case, we can assume 1024 cross correlations and 8 self-correlations per baseline, 351 baselines, and 10s integrating time. This implies 27.6 μs average processing time per correlation. If the processing is completely pipelined, then all 27.6 μs is available for the quantization correction, otherwise less time is available. It will probably be somewhat difficult to implement the calculations in a minicomputer at this speed; the burden would be considerably eased if the maximum number of cross-correlations per baseline were reduced to 512.

In any case, I shall assume that after correction we have, for each cross-correlation measured, a fixed-point, twos-complement word of length b* bits, with the binary point just after the sign bit, representing the cross-correlation coefficient ρ_{ij}.

4. Averaging Pairs of Measurements

For some observations it is desirable to be able to average together pairs of correlation measurements before proceding with further processing, thus cutting the amount of data in half. In fact, for any modes which produce 1024 correlations per baseline, either such averaging or deletion of data is expected to be required, lest the capacity of the synchronous and asynchronous computers be exceeded. The main use of such averaging is expected to be the combining of measurements made at orthogonal polarizations in order to improve the signal-to-noise ratio in total-power maps of low-polarization sources.

Pair averaging must be done after the quantization correction. After a pair of numbers is added, the result should be divided by 2 (right shifted one bit) to maintain the same scaling as before; if proper allowance is made for overflow in case $|\rho|>0.5$, then no loss of precision results.

5. Fast Fourier Transform

The resulting vector of cross-correlation coefficients for each baseline may now be transformed in an FFT processor. Numerical errors in such processors have been extensively analyzed in the literature (e.g., James 1975, Welch 1969) and will not be discussed here. In general, processors which do all arithmetic in floating point form maintain the best precision for a given word length, but are slower for a given cost. In fixed-point processors, various algorithms can be employed to re-scale the data before each stage of the FFT in order to minimize roundoff errors; careful consideration should be given to the algorithm employed by any processor proposed for purchase.

6. Additional Processing

Several other operations are necessary on the FFT output vectors and will be necessary before beginning the map-making process. These can be expected to be shared between the correlator and synchronous computer subsystems as resources permit. Some of the more obvious operations are:

(a) Multiplication by $\sqrt{T_{si}\ T_{sj}}$, where T_{si} is the measured system temperature of the ith antenna, thus correcting for variations in receiver and ALC gains.

(b) Correction for variation of complex gain across the passband, if this can be measured independent of astronomical calibration.

(c) Application of astronomical calibration data.

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