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SIGNAL-TO-NOISE RATIO IN DIGITAL CROSS-CORRELATORS AS A FUNCTION OF CHANNEL NUMBER

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1.0 INTRODUCTION

A digital correlator is a device for measuring the correlation function of a signal or signals which are sampled and quantized. For gaussian noise signals, the quantization can be very coarse with little loss of information; the literature includes extensive analyses of correlators using two-level (one bit) [Weinreb, 1961], and three or four level (two bit) [Cooper, 1970] quantizations. In addition, the effect of varying the sampling rate has been studied [Hagen and Farley, 1973], showing that significant improvements in signal-to-noise ratio can be obtained for ideal low-pass signals of bandwidth B by sampling at rates greater than 2B. This is because the quantization causes the signal to be no longer bandlimited, so not all information is extracted by sampling at only 2B.

The published results, however, consider only the accuracy with which a single correlation measurement can be made for signals having ideal low-pass spectra. In this report, I analyze the correlator performance for non-ideal signal spectra, such as might be imposed by practical filters at the correlator inputs. Furthermore, I consider that the correlation function will be measured at a large but finite number of equally-spaced lags, or delays, and that the discrete Fourier transform of this sequence of measurements will be computed. The signal-to-noise ratio (SNR) in each element, or channel, of the transform is then of interest. This quantity is computed in Section II. The computation is restricted to cross-correlators, and to the case where the cross-correlation coefficient is small at all lags.

In Section III, the results are evaluated for some practical input filters, with the objective of determining the filter shapes and cutoff frequencies which will maximize the number of high-SNR channels.

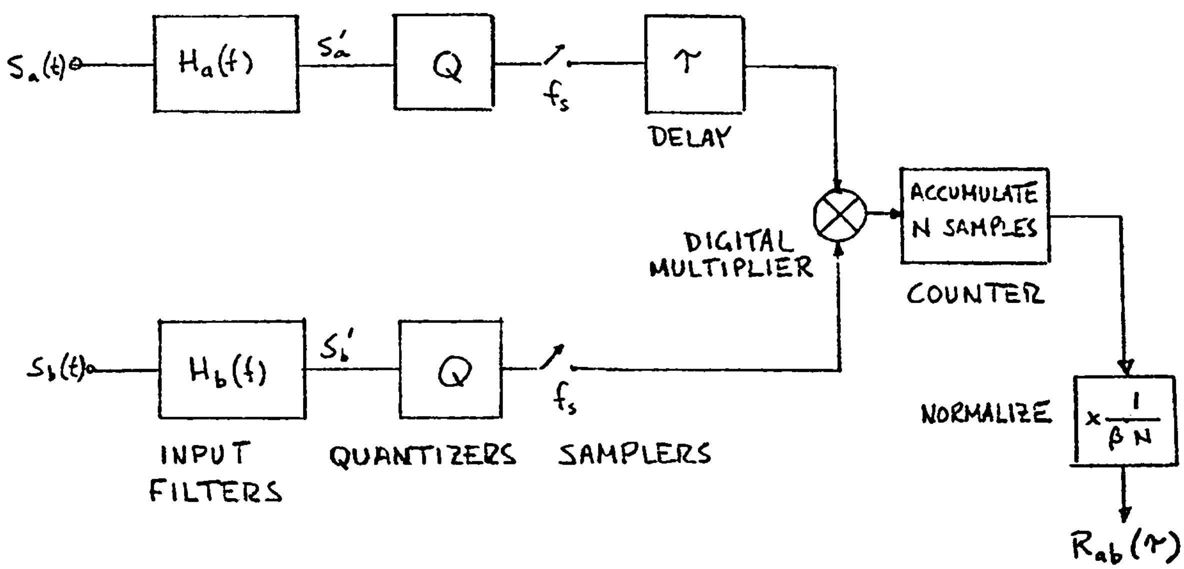

Figure 1 - Correlator Block Diagram

II. CALCULATIONS

Consider the correlator of Figure 1. Suppose that we use it to compute $R_{ab}(\tau)$ for $\tau = -Kd/f_s, -K(d-1)/f_s, \ldots, 0, d/f_s, 2d/f_s, \ldots (K-1)d/f_s$; that is, we compute

$$R_{ab}(kd/f_s) = \frac{1}{\beta N} \sum_{i=1}^{N} a_{i-kd} b_i, \quad k = -K, \ldots, 0, \ldots, K-1 \tag{1}$$

where βN is the count obtained for fully correlated signals (if Q is a three-level quantizer with optimum thresholds, then $\beta = 0.54$), and d is the delay step in multiples of the sampling interval. Next, suppose that we compute the length-2K discrete Fourier transform (DFT) of this sequence, obtaining

$$S_{ab}(\ell) = \sum_{k=-K}^{K-1} R_{ab}(kd/f_s) e^{-j2\pi k\ell/2K}, \quad \ell = 0, \ldots, K-1. \tag{2}$$

We are interested in the SNR at each discrete frequency ℓ $(=fKd/f_s)$; that is, in

$$\mathrm{SNR}(\ell) \equiv \frac{\langle S_{ab}(\ell) \rangle}{[\mathrm{var}\left(S_{ab}(\ell)\right)]^{1/2}} \tag{3}$$

and in particular we would like to know the effect of the input filters on this SNR.

Assume that $s_a(t)$ and $s_b(t)$ are white Gaussian noise processes with autocorrelation functions $\delta(\tau)$ and cross-correlation function $\rho_o\delta(\tau)$ with $\rho_o << 1$. Then the corresponding functions after filtering are

$$\rho'_{aa}(\tau) = \int_{-\infty}^{\infty} |H_a(f)|^2 \, e^{j2\pi f\tau} df \tag{4a}$$

$$\rho'_{bb}(\tau) = \int_{-\infty}^{\infty} |H_b(f)|^2 \, e^{j2\pi f\tau} df \tag{4b}$$

$$\rho'_{ab}(\tau) = \rho_o \int_{-\infty}^{\infty} H_a(f) H_b^*(f) e^{j2\pi f\tau} df \tag{4c}$$

Without loss of generality, we can assume that the filter gains are normalized so that $\int |H_{a,b}(f)|^2 df = 1$; then ρ'_{aa}, ρ'_{bb}, and ρ'_{ab} are also correlation coefficients.

For all commonly-considered quantizations Q (one-bit, two-bit, three-level, etc.), $\langle R_{ab}(\tau)\rangle$ is simply proportional to $\rho'_{ab}(\tau)$ provided that the latter is small (as is the case here since we have assumed $\rho_o \ll 1$). Let α be the proportionality constant, so

$$\langle R_{ab}(\tau)\rangle = \alpha \rho'_{ab}(\tau) \tag{5}$$

(for the optimum three-level quantizer, $\alpha = 0.81$); then from (2) and (4c)

$$\langle S_{ab}(\ell)\rangle = \alpha \sum_{k=-k}^{K-1} \rho'_{ab}(dk/f_s) e^{-j2\pi k\ell/2K}$$

$$= \alpha\rho_o \sum_{k=-K}^{K-1} \int H_a(f) H_b{}^*(f) e^{-j2\pi k(\ell/2K - fd/f_s)} df. \tag{6}$$

We next calculate the variance of $S_{ab}(\ell)$ directly from its definition:

$$\text{var } S_{ab}(\ell) = \langle |S_{ab}(\ell)|^2\rangle - \langle S_{ab}(\ell)\rangle^2 . \tag{7}$$

In view of the assumption that $\rho_o \ll 1$, we neglect the second term in (7); this can be regarded as a calculation of the noise under no-signal conditions. Substituting directly from (1) and (2) into (7) gives

$$\text{var } S_{ab}(\ell) = \langle \left| \frac{1}{\beta N} \sum_{k=-K}^{K-1} \sum_{i=1}^{N} a_i b_{i-dk} \; e^{-j2\pi k\ell/2K} \right|^2 \rangle$$

$$= \frac{1}{\beta^2 N^2} \sum_{k=-K}^{K-1} \sum_{i=1}^{N} \sum_{k'=-K}^{K-1} \sum_{i'=1}^{N} \langle a_i b_{i-dk} a_{i'} b_{i'-dk'} \rangle e^{-j2\pi k\ell/2K}$$

$$= \frac{1}{N^2} \sum_k \sum_i \sum_{k'} \sum_{i'} R_{aa}(i-i') R_{bb}(i-dk-i'+dk') e^{-j2\pi(k-k')\ell/2K} \tag{8}$$

where we have used, in the last step,

$$R_{aa}(i-i') \equiv \frac{1}{\beta} \langle a_i a_{i'} \rangle \tag{9a}$$

$$R_{bb}(j-j') \equiv \frac{1}{\beta} \langle b_j b_{j'} \rangle . \tag{9b}$$

Note that the normalization ensures that $R_{aa}(0) = R_{bb}(0) = 1$. Making a change of variables in (8) by letting $m = i-i'$ and $m = k-k'$ gives

$$\mathrm{var}\ S_{ab}(\ell) = \frac{1}{N^2} \sum_{k=-K}^{K-1} \sum_{i=1}^{N} \sum_{n=k-K+1}^{k+K} \left[\sum_{m=i-N}^{i} R_{aa}(m) R_{bb}(m-dn) \right] e^{-j2\pi n\ell/2K} . \tag{10}$$

Now, if the N samples take many reciprocal bandwidths to accumulate, then $N \gg$ (width of R_{aa} or R_{bb}); then, for almost all values of i, the bracketed sum extends over all significant values of its argument. We may therefore extend the limits of that sum to infinity with little error. This also allows evaluating the sum over i as a multiplication by N, giving

$$\mathrm{var}\ S_{ab}(\ell) = \frac{1}{N} \sum_{k=-K}^{k-1} \sum_{n=k-K+1}^{k+K} e^{-j2\pi n\ell/2K} \sum_{m=-\infty}^{\infty} R_{aa}(m) R_{bb}(m-dn) . \tag{11}$$

Next, since k appears only in the limits of the sum over n, the outer two sums can be combined. We note that n takes on the values $-2K+1, -2K+2, \ldots, 0, 1, \ldots 2K-1$, and that the value $n=i$ occurs $2K-|i|$ times. Thus,

$$\mathrm{var}\ S_{ab}(\ell) = \frac{1}{N} \sum_{i=-2K+1}^{2K-1} (2K-|i|) e^{-j2\pi i\ell/2K} \sum_{m=-\infty}^{\infty} R_{aa}(m) R_{bb}(m-di) . \tag{12}$$

We now relate this to the filter transfer functions H_a, H_b by noting that R_{aa}, R_{bb} are fixed, monotonic functions of ρ'_{aa}, ρ'_{bb}, respectively, where the function depends only on the kind of quantization used (for 2-bit quantization and its special cases, the functions are tabulated by Cooper, 1970). Let the function be $C_Q(\cdot)$, so that

$$R_{aa}(m) = C_Q\left(\rho'_{aa}(m/f_s)\right) \tag{13a}$$

$$R_{bb}(n) = C_Q\left(\rho'_{bb}(n/f_s)\right). \tag{13b}$$

Substituting this in (12) gives

$$\text{var } S_{ab}(\ell) = \frac{1}{N} \sum_{i=-2K+1}^{2K-1} (2K-|i|)e^{-j2\pi i\ell/2K} \times \sum_{m=-\infty}^{\infty} C_Q\left(\rho'_{aa}(m/f_s)\right) C_Q\left(\rho'_{bb}(m-di/f_s)\right) \tag{14}$$

and using (14) and (6) in (3) gives the desired expression for SNR:

$$\text{SNR}(\ell) = \alpha\sqrt{\frac{N}{2K}} \frac{\sum_{k=-K}^{K} \rho'_{ab}(k/f_s)e^{-j2\pi k\ell/2K}}{\left[\sum_{i=-2K+1}^{2K-1} (1-\frac{|i|}{2K})e^{-j2\pi k\ell/2K} \sum_{m=-\infty}^{\infty} C_Q\left(\rho'_{aa}(m/f_s)\right) C_Q\left(\rho'_{bb}(m-di/f_s)\right)\right]^{1/2}}. \tag{15}$$

Note that ρ'_{ab}, ρ'_{aa}, and ρ'_{bb} in (15) are given in (4) and are simply inverse Fourier integrals involving the filter transfer functions. We have thus established the relationship between the filters and the SNR.

Equation (15) apparently cannot be further simplified without making approximations. However, the following approximation can usually be justified: Let $R_{aa}(m), R_{bb}(m) = 0$ for $|m|>Kd$. If the filters approximate ideal low-pass functions, then R_{aa}, R_{bb} approximate sinc functions, which are small for large values of their arguments. If we also take $d=1$, then (12) may be reduced by interchanging the summations and substituting $n=m-i$:

$$\text{var } S_{ab}(\ell) = \frac{1}{N}\sum_{m=-\infty}^{\infty} R_{aa}(m) \sum_{n=m+2K-1}^{m-2K+1} (2K-|n-m|)R_{bb}(n)e^{-j2\pi(n-m)\ell/2K}$$

$$= \frac{1}{N}\sum_{m=-K}^{K-1} R_{aa}(m)e^{j2\pi m\ell/2K} \sum_{m=-k}^{K-1} (2K-|n-m|)R_{bb}(n)e^{-j2\pi n\ell/2K} \qquad (16)$$

Making the further arrpoximation of dropping the $|n-m|$ term (since it multiplies only terms whose magnitude is relatively small), this may be written

$$\text{var } S_{ab}(\ell) = \frac{2K}{N} S_{aa}(\ell)S_{bb}(\ell) \qquad (17)$$

where S_{aa} is the DFT of R_{aa}, etc.
The simplified expression for SNR thus becomes

$$\text{SNR}(\ell) = \alpha\sqrt{\frac{N}{2K}} \frac{\text{DFT}\{\rho'_{ab}(k/f_s)\}}{\text{DFT*}\{C_Q\left(\rho'_{aa}(k/f_s)\right)\}\text{DFT}\{C_Q\left(\rho'_{bb}(k/f_s)\right)\}^{\frac{1}{2}}}. \qquad (18)$$

Finally, with identical input filters $H_a(f)\equiv H_b(f)$, we see from (4) that $\rho'_{aa}=\rho'_{bb}=\rho'_{ab}/\rho_o\equiv\rho'$, so in that case

$$\text{SNR}(\ell) = \alpha\rho_o\sqrt{\frac{N}{2K}} \frac{\text{DFT}\{\rho'(k/f_s)\}}{\text{DFT}\{C_Q\left(\rho'(k/f_s)\right)\}}, \quad \ell=0,1,\ldots,K. \qquad (19)$$

III. APPLICATION

Expression (19) has been evaluated numerically for filter designs of the Chebychev family having various parameters, as listed in Table 1.

Table 1: Filters Evaluated

Type	Ripple Parameter	Number of Elements	Bandwidth Range
Butterworth	-	6	$0.3f_s$ to $0.5\ f_s$
Butterworth	-	8	"
Chebychev	0.1 dB	6	"
Chebychev	0.1 dB	8	"

Only low-pass filters were considered. $C_Q(\cdot)$ was taken from the tabulated function [Cooper, 1970] for 3-level quantization and 3-level products, as used in the VLA correlator. The DFT length was $K = 128$.

Figure 2 is a plot of SNR(ℓ) for each of the four filter types, with the bandwidth choosen in each case for 20 dB rejection at $f = 0.55\ f_s$ (10% above Nyquist bandwidth). Figure 3 is a plot of the fraction of channels for which the SNR exceeds 0.8 of its "ideal" value of $\alpha\rho_o\sqrt{N/2K}$, as a function of filter bandwidth.

IV. DISCUSSION

Near the highest channel, which corresponds to frequency $f = f_s/2$, the SNR is degraded by two effects: the quantization non-linearity causes noise to be scattered from the lower frequencies upward, and the sampling causes noise to be aliased from the higher frequencies (including both noise passed by the finite filter rejection and the scattered noise just mentioned) downward. Nevertheless, Figure 3 shows that it is easily possible to have all channels with less than 1 dB SNR "loss" from these effects. But other effects may prevent this: the filter must have sufficient rejection for $f > f_s/2$ to avoid aliasing of correlated signals into the output channels; we have not regarded such undesired signals as "noise" in this calculation. Further discussion of this and other compromises required in selecting practical filters will be given in a later report.

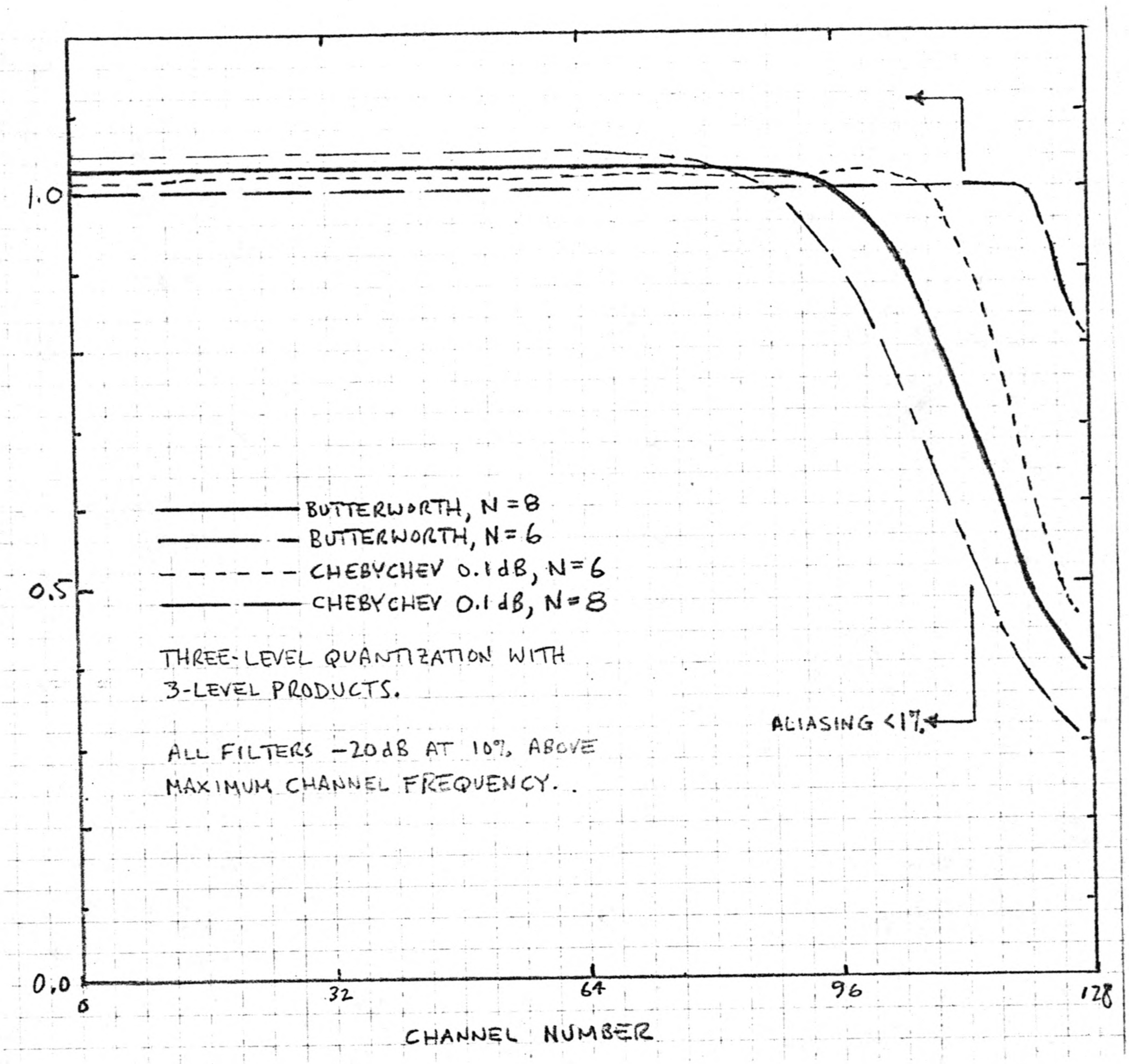

Figure 2 - Signal-to-noise ratio vs. channel number for four different input filters. The filters are assumed identical for both signals. The ordinate is in units of the "ideal" SNR, $\alpha\rho_o\sqrt{N/2K}$. All filters cut off to -20 dB at 10% above the maximum channel frequency, so that aliased signals are below 1% over 90% of the channels.

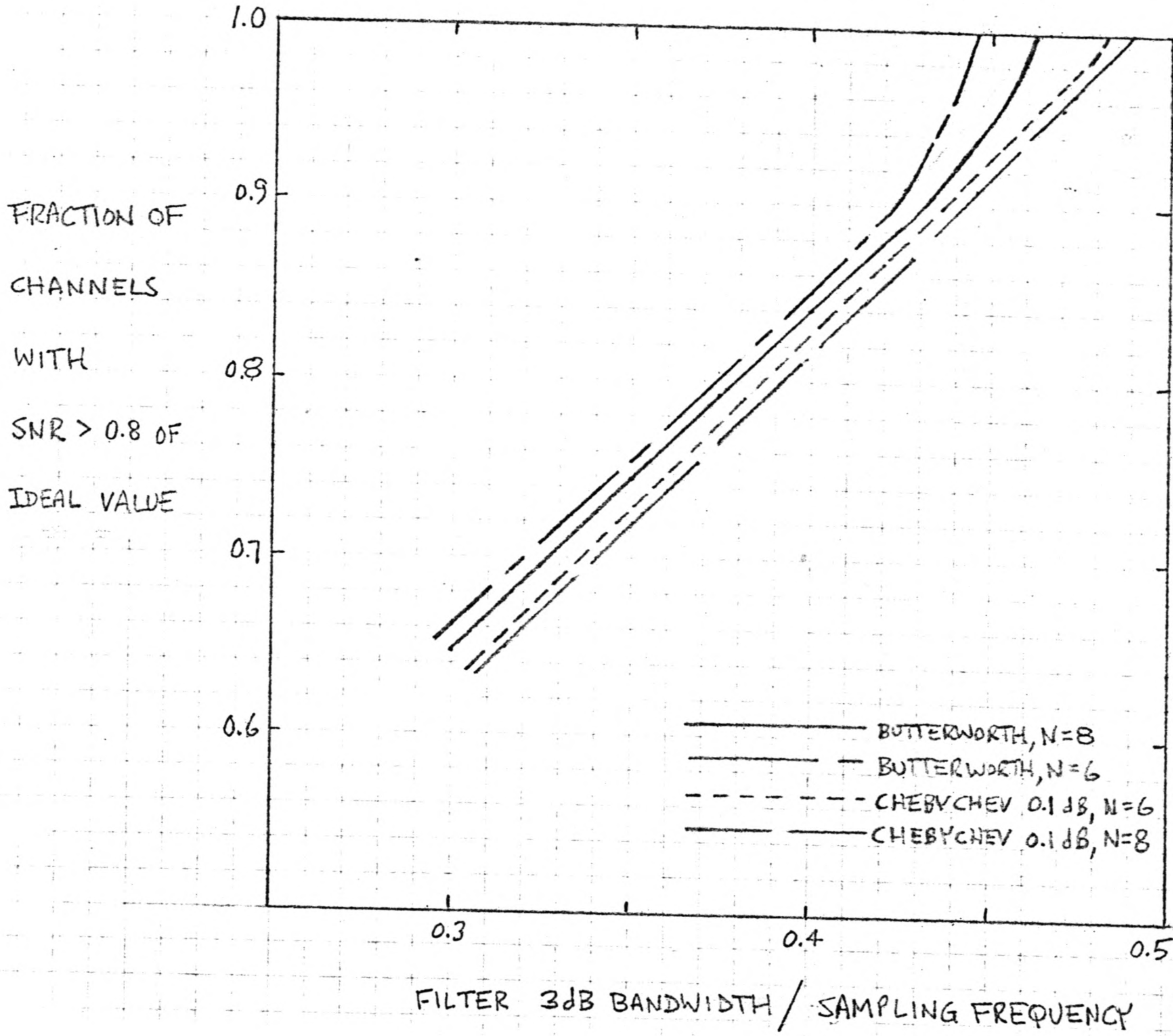

Figure 3 - Fraction of channels for which SNR > 0.8 $\alpha\rho_o\sqrt{N/2K}$ as a function of filter bandwidth, for each of four filters.

V. REFERENCES

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