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FORMERLY WILLOW RUN LABORATORIES, THE UNIVERSITY OF MICHIGAN

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MEMORANDUM TO: VLA Optical Processor File

FROM: James R. Fienup J. F.

SUBJECT: Detected Signal

The purpose of this memo is to account for the various terms present when two images are subtracted that were formed using reference wavefronts of relative phase zero and pi radians, respectively. Also discussed is the effect of reconstructing only half of the u-v plane.

CASE 1: Exact Fourier transform (input plane in front of transforming lens). The reference wavefront is given by a plane wave

$$E_r(x,y) = r_o e^{jkx}$$

where $r_o = r_o' + jr_o''$ is a complex constant, with real and imaginary components r_o' and r_o'', respectively.

The signal wavefront is given by

$$E_s(x,y) = E_s'(x,y) + jE_s''(x,y)$$

where E_s' and E_s'' are the real and imaginary components, respectively. $E_s'(x,y)$ consists of the desired brightness distribution, $B(x,y)$ a real-valued function plus the real part of the error due to system aberrations. $E_s''(x,y)$ consists of the imaginary part of the error due to system aberrations. The detected intensity is

$$I_1(x,y) = |E_r(x,y) + E_s(x,y)|^2$$

$$= |r_o e^{jkx} + E_s|^2$$

$$= |r_o|^2 + |E_s|^2 + r_o e^{jkx} E_s^* + r_o^* e^{-jkx} E_s$$

$$= |r_o|^2 + |E_s|^2 + \left[(r_o'E_s' + r_o''E_s'') + j(-r_o'E_s'' + r_o''E_s')\right] e^{jkx}$$

$$+ \left[(r_o'E_s' + r_o''E_s'') + j(r_o'E_s'' - r_o''E_s')\right] e^{-jkx}$$

$$= |r_o|^2 + |E_s|^2 + 2(r_o'E_s' + r_o''E_s'')\cos kx + 2(r_o'E_s'' - r_o''E_s')\sin kx$$

If the reference beam is shifted by pi radians, then

$$I_2(x,y) = |r_o|^2+|E_s|^2-2(r_o'E_s'+r_o''E_s'')\cos kx -2(r_o'E_s''-r_o''E_s')\sin kx$$

The difference, then, is

$$\Delta I(x,y) = I_1-I_2 = 4(r_o'E_s'+r_o''E_s'')\cos kx +4(r_o'E_s''-r_o''E_s')\sin kx$$

For a reference wave adjusted to a relative phase of zero, $r_o'' = 0$ and the equation above reduces to

$$\Delta I(x,y) = 4\ r_o'E_s'\ \cos kx + 4\ r_o'E_s''\ \sin kx$$

which should allow the separation of the term E_s'' from the desired term. For example, one might sample $\Delta I(x,y)$ only when $\sin kx = 0$. For an on-axis plane wave reference beam, the above further simplifies to

$$\Delta I(x,y) = 4\ r_o'E_s' ,$$

completely eliminating the term E_s''. However, E_s' still contains the real component of the error due to aberrations, which cannot be distinguished from B(x,y).

CASE 2: Fourier transform with quadratic phase (input plane behind transforming lens). In the paraxial case, a spherical reference wavefront offset in the x-direction is approximated by

$$E_r(x,y) = r_o \exp\left\{ j\frac{\pi}{\lambda z}\left[(x-x_o)^2 + y^2\right]\right\}$$

$$= \left[r_o \exp\left\{ j\frac{\pi x_o^2}{\lambda z}\right\}\right]\exp\left\{ j\frac{\pi}{\lambda z}(x^2+y^2)\right\} \exp\left\{ -j\frac{\pi x_o}{\lambda z} x\right\}$$

And the Fresnel transform gives the signal wavefront

$$e^{j\frac{\pi}{\lambda z}(x^2+y^2)} E_s(x,y) = e^{j\frac{\pi}{\lambda z}(x^2+y^2)}\left[E_s'(x,y) + j\, E_s''(x,y)\right]$$

Then

$$I_1(x,y) = \left| r_o e^{jk_o x} \exp\left\{ j\frac{\pi}{\lambda z}(x^2+y^2)\right\} + E_s(x,y)\exp\left\{ j\frac{\pi}{\lambda z}(x^2+y^2)\right\}\right|^2$$

$$= \left| r_o e^{jk_o x} + E_s(x,y)\right|^2$$

where the factor $\exp\left\{\frac{\pi x_o^2}{\lambda z}\right\}$ was absorbed into r_o and $k_o = -\frac{\pi x_o}{\lambda z}$. Thus, we see that to within the accuracy of the Fresnel approximation, the quadratic phase terms drop out, leaving the same intensity as in Case 1.

CASE 3: Non-paraxial case (input plane behind transforming lens). In the non-paraxial case, a spherical reference wavefront is given by

$$E_r(x,y) = \frac{r_o \exp\left\{ j\frac{2\pi}{\lambda} \sqrt{z^2 + (x-x_o)^2 + y^2} \right.}{\sqrt{z^2 + (x-x_o)^2 + y^2}}$$

The wavefront passing through a transparency of amplitude transmittance T(u,v) in the (u,v) plane illuminated by a converging spherical wavefront with the origin of the output plane as its center is, according to the Huygens-Fresnel principle (see Goodman, Introduction to Fourier Optics, p. 58),

$$E_s(x,y) = \frac{1}{j\lambda} \iint T(u,v) e^{jW(u,v)} \frac{\exp\left\{ j\frac{2\pi}{\lambda}(r_{ux}-r)\right\}}{r_{ux} r} \left(\frac{z}{r_{ux}}\right) du\, dv$$

where $r_{ux} = \sqrt{z^2 + (u-x)^2 + (v-y)^2}$, $r = \sqrt{z^2 + u^2 + v^2}$, (z/r_{ux}) is the obliquity factor, and W(u.v) includes aberrations in the wavefront illuminating the transparency. Note that in this formulation, W(u,v) is independent of x and y. In this case, aberration terms having an x-y dependence arise from the expansion of r_{ux}. Rewriting r_{ux}, expanding about z, using the expansion $(1+x)^n = 1 + nx + \frac{n(n-1)}{2!} x^2 + \ldots$, we have

$$r_{ux} = \sqrt{z^2 + (u-x)^2 + (v-y)^2} = \sqrt{z^2 + (u^2 + v^2 - 2ux - 2vy) + (x^2 + y^2)}$$

$$= z\left\{1 + \frac{1}{2z^2}\left[(u^2 + v^2 - 2ux - 2vy) + (x^2 + y^2)\right]\right.$$

$$\left. - \frac{1}{8z^4}\left[(u^2 + v^2 - 2ux - 2vy) + (x^2 + y^2)\right]^2 + \ldots\right.$$

and

$$r = \sqrt{z^2 + u^2 + v^2} = z\left[1 + \frac{1}{2z^2}(u^2 + v^2)^2 + \ldots\right]$$

The terms in $(r_{ux}-r)$ that are independent of u and v that can be brought out of the integral are $z\left[1 + \frac{1}{2z^2}(x^2+y^2) - \frac{1}{8z^4}(x^2+y^2)^2 + \ldots\right] - z = \sqrt{z^2+x^2+y^2} - z$

That is, $E_s(x,y)$ will have the phase of a spherical wavefront associated with it. As in Case 2, we redefine $E_s(x,y)$ to separate out this phase term in order to determine its effect on the detected image:

$$I_1(x,y) = \left| \frac{r_o \exp\left\{ j\frac{2\pi}{\lambda}\sqrt{z^2+(x-x_o)^2+y^2} \right\}}{\sqrt{z^2+(x-x_o)^2+y^2}} + \exp\left\{ j\frac{2\pi}{\lambda}\sqrt{z^2+x^2+y^2} \right\} E_s(x,y) \right|^2$$

$$= |r_o|^2 / \left[z^2 + (x-x_o)^2 + y^2 \right] + |E_s(x,y)|^2$$

$$+ \exp\left\{ j\frac{2\pi}{\lambda} \left[\sqrt{z^2 + (x-x_o)^2 + y^2} - \sqrt{z^2 + x^2 + y^2} \right] \right\}.$$

$$\cdot \left[(r_o' E_s' + r_o'' E_s'') + j\,(-r_o' E_s'' + r_o'' E_s') \right] / \sqrt{z^2 + (x-x_o)^2 + y^2}$$

$$+ \text{(complex conjugate)}$$

$$= |r_o|^2 / \left[z^2 + (x-x_o)^2 + y^2 \right] + |\, E_s(x,y)|^2$$

$$+ \frac{2(r_o'E_s'+r_o''E_s'')}{\sqrt{z^2+(x-x_o)^2+y^2}} \cos\left[\frac{2\pi}{\lambda}\left(\sqrt{z^2+(x-x_o)^2+y^2} - \sqrt{z^2+x^2+y^2}\right)\right]$$

$$+ \frac{2(r_o'E_s''-r_o''E_s')}{\sqrt{z^2+(x-x_o)^2+y^2}} \sin\left[\frac{2\pi}{\lambda}\left(\sqrt{z^2+(x-x_o)^2+y^2} - \sqrt{z^2+x^2+y^2}\right)\right]$$

A reference beam shifted by π radians causes the last two terms in the expression above to change sign. Taking the difference and adjusting the reference beam to a relative phase of zero ($r_o'' = 0$) yields

$$\Delta I(x,y) = \frac{4r_o'E_s'(x,y)}{\sqrt{z^2+(x-x_o)^2+y^2}} \cos\left[\frac{2\pi}{\lambda}(\sqrt{z^2+(x-x_o)^2+y^2} - \sqrt{z^2+x^2+y^2}\right]$$

$$+ \frac{4r_o'E_s''(x,y)}{\sqrt{z^2+(x-x_o)^2+y^2}} \sin\left[\frac{2\pi}{\lambda}(\sqrt{z^2+(x-x_o)^2+y^2} - \sqrt{z^2+x^2+y^2}\right]$$

which should allow the separation of the term E_s'' from the term E_s'.

This separation, however, would be more difficult than the corresponding separation in Case 1, due to the more complicated argument of the sine and cosine terms. For an on-axis point-source reference wave, however, this simplifies to

$$\Delta I(x,y) = 4r_o'E_s'(x,y)$$

completely eliminating the term E_s''. However, E_s' still contains not only the real component of the error due to aberrations of the illuminating wavefront and of the signal transparency, but also due to the discrepancy between the Fresnel approximation (Case 2) and the more exact formulation (Case 3).

Reconstruction of Half of the u-v Plane

Let $B(x,y) = B'(x,y) + j\,B''(x,y)$ be the Fourier transform of $V(u,v) = V'(u,v) + j\,V''(u,v)$:

$$B(x,y) = B'(x,y) + j\,B''(x,y) = \iint V(u,v)\ \exp\left\{ j\frac{2\pi}{\lambda f}(ux+vy)\right\} du\ dv$$

Equating the real parts and imaginary parts, we find that

$$B'(x,y) = \iint_{-\infty}^{\infty} \left[V'(u,v)\cos\frac{2\pi}{\lambda f}(ux+vy) - V''(u,v)\sin\frac{2\pi}{\lambda f}(ux+vy)\right] du\ dv$$

and

$$B''(x,y) = \iint_{-\infty}^{\infty} \left[V''(u,v)\cos \frac{2\pi}{\lambda f}(ux+vy) + V'(u,v)\sin \frac{2\pi}{\lambda f}(ux+vy) \right] du\ dv$$

Therefore, if $B''(x,y) = 0$, then $V''(u,v)$ must be odd and $V'(u,v)$ even; that is, if $B(x,y)$ is real, then $V(u,v)$ is Hermetian: $V(-u,-v) = V^*(u,v)$. Now suppose we reconstruct only half of the u-v plane to obtain

$$B_h(x,y) = B_h'(x,y) + j\ B_h''(x,y)$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} V\ (u,v)\exp\left\{ j\frac{2\pi}{\lambda f}(ux+vy) \right\} du\ dv$$

$$= \int_{-\infty}^{\infty}\int_{0}^{\infty} \left[V'(u,v)\cos \frac{2\pi}{\lambda f}(ux+vy) - V''(u,v)\sin \frac{2\pi}{\lambda f}(ux+vy) \right] du\ dv$$

$$+\ j \int_{-\infty}^{\infty}\int_{0}^{\infty} \left[V''(u,v)\cos \frac{2\pi}{\lambda f}(ux+vy) + V'(u,v)\sin \frac{2\pi}{\lambda f}(ux+vy) \right] du\ dv$$

Since both terms in the first intergrand are even, the integral over only half the u-v plane is equal to one half of the integral over the entire u-v plane. Consequently, the first integral is equal to

$$B_h'(x,y) = \frac{1}{2} B'(x,y) = \frac{1}{2} B(x,y)$$

The second integral equal to $jB''_h(x,y)$, which is purely imaginary, can be considered to be noise. Another way of looking at the half-plane reconstruction is that we are multiplying $V(u,v)$ by the step function $\{1, u>0; 0, u<0\}$. Thus the image is convolved with the Fourier transform of the step function, $\frac{1}{2}\delta(x)-j/2\pi x$, resulting in a complex amplitude that is $\frac{1}{2}B(x,y)$ plus an imaginary-valued term.

Note that if we can detect the real part of the image apart from the imaginary part, as described in Cases 1-3 above, then we can obtain the desired brightness distribution $B(x,y)$ with no loss of resolution despite the fact that our aperture is only half the original width. This phenomena occurs only when the desired image is of constant phase (e.g., real-valued), so that the term arising from the convolution with $-j/2\pi x$ can be separated out from the term arising from the convolution with $\frac{1}{2}\delta(x)$; equivalently, the data in the u-v plane must be Hermetian. A partial explanation of this phenomenon is that since the addition of the second half-plane of a Hermetian function adds no new information that cannot be obtained from the first half-plane, then there also should be no additional information in the image (such as would be obtained if the resolution were doubled).

The desired image being obtainable with no loss in resolution, the reconstruction of only half the u-v plane is attractive since it reduces the space-bandwidth-product requirement on the optical recording device by a factor of two; however, signal-to-noise consideration reduce its attractiveness. Consider the aberration term $W(u,v)$ in the integral equation for $E_s(x,y)$ under Case 3 above. For small aberrations, $\exp\left[jW(u,v)\right] \simeq 1 + jW(u,v)$. To within the limits of the Fresnel approximation, the image $B_h(x,y)$

is convolved with $\delta(x,y) + j\,\mathcal{F}\{W(u,v)\}$, where $\mathcal{F}\{W(u,v)\}$ is the Fourier transform of $W(u,v)$. For $W(u,v)$ even, $j\,\mathcal{F}\{W(u,v)\}$ is imaginary, and for $W(u,v)$ odd, $j\,\mathcal{F}\{W(u,v)\}$ is real-valued. Thus, the desired image is convolved with an error term $j\mathcal{F}\{W(u,v)\}$ that has both real and imaginary components. When the full u-v plane is used, the ideal image $B(x,y)$ is purely real, and the convolution of $B(x,y)$ with the imaginary component of the error term is imaginary and can be separated from the desired term. Only the convolution of $B(x,y)$ with the real part of the error term is kept. Thus the real-value detection process causes an increase in signal-to-noise ratio. When only half the u-v plane is used, then the ideal image has both a real component, $B_h'(x,y)$ and an imaginary component, $B_h''(x,y)$; and we would expect these two components to have approximately equal power. When these two terms are convolved with the error term, there result two real terms: the convolution of $B_h'(x,y)$ with the real component of the error term and the convolution of $B_h''(x,y)$ with the imaginary component of the error term. Thus, the signal-to-noise ratio obtained when using half the u-v plane is approximately half that obtained when using the full u-v plane.

JRF:sd

cc: C. Aleksoff
M. Carter
I. Cindrich
M. Hidayet
A. Klooster