3 August 1976

MEMORANDUM TO: VLA Optical Processor File

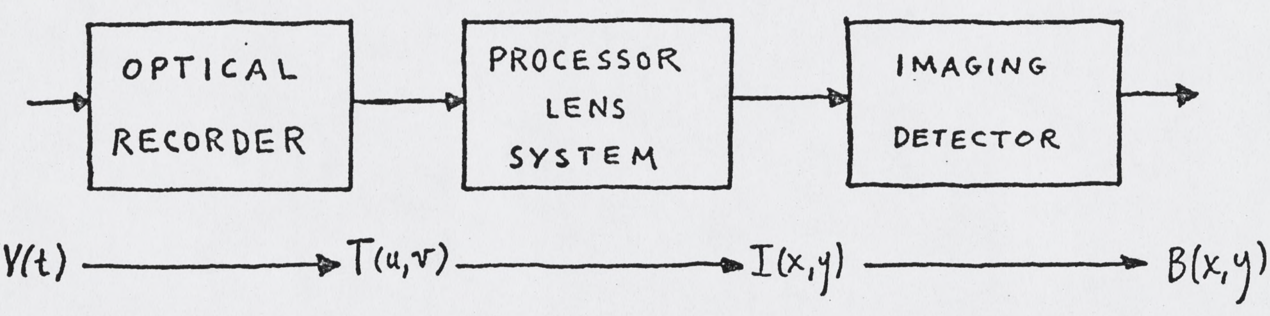
FROM: M. Carter

SUBJECT: Linear Modeling of the VLA Optical Processor System

This memo describes the development of a linear model for the VLA optical processor system, and subsequent characterization of the elements of the model. It is hoped that the model will provide a basis for obtaining a quantitative measure of system performance, at least to a reasonable approximation. Some general observations on the behavior of the model are included.

SYSTEM NON-IDEALITIES

The VLA optical processor system is comprised of three major subsystems as shown below:



The input to the system is the correlator signal V(t), which corresponds to the complex visibility function V(u,v). The optical recorder produces a two-dimensional spatial recording of the visibility function, $T(u,v)$, which is the input to the processor lens system. The output is a light intensity distribution containing a term proportional to the sky brightness map $B(x,y)$.

In developing the system model, it is necessary to examine non-idealities that are present in each of the sub-system blocks. Strictly speaking, the system is not linear since the detector is a square-law device. However, by addition of a reference beam, we may create an intensity distribution containing a term proportional to the desired brightness map. This procedure effectively "linearizes" the system. Possible error sources include:

Optical Recorder

- writing beam position errors
- electronics noise (in amplifiers, etc.)
- lowpass behavior of recording film
- film-grain noise

Processor Lens System

- aberration effects in Fourier transform channel (includes transform lens, liquid gate, and optical path.)
- scattering noise
- aberrations in reference beam-forming channel (includes beam forming lens and optical path.)
- accuracy of π phase shifter for reference wave.

Imaging Detector

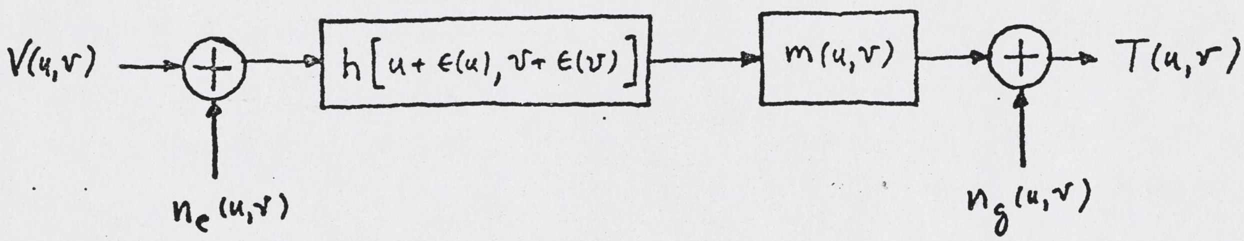
- finite detector array resolution
- non-uniform spatial responsivity
- thermal and shot noises
- array positioning errors
- quantization noise in output A/D conversion

The actual physical configuration of the processing system is assumed to be that described by Bulabois (1). It is the most simple of those presented, and thus is a good candidate for representation by a linear model.

PROPOSED MODEL

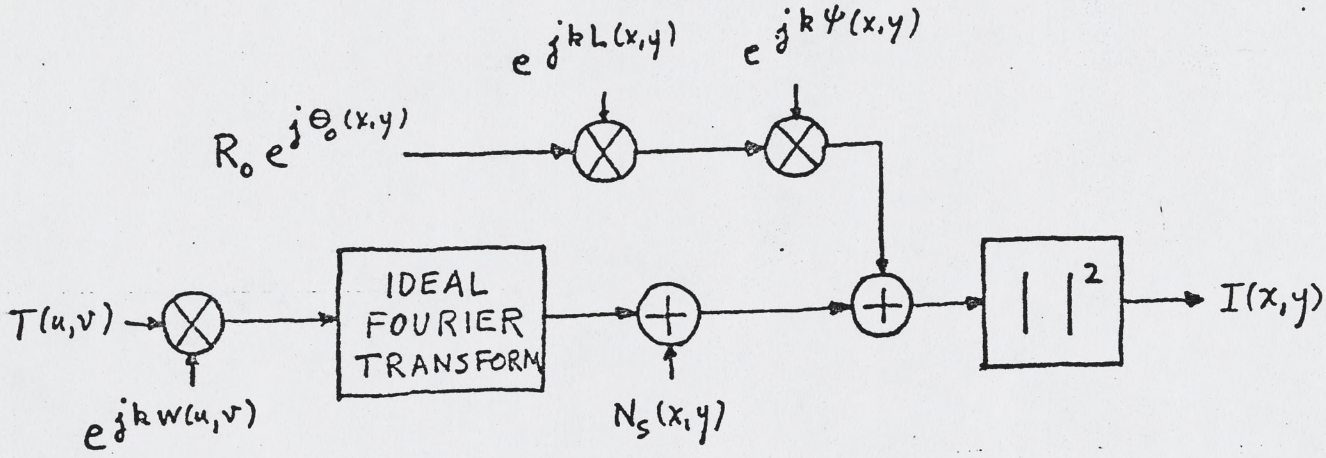
We treat separately the modeling of each subsystem. Characterization of the model elements will be discussed in the next section.

A. Optical Recorder Model



$V(u,v)$	-	complex visibility data
$n_e(u,v)$	-	electronics noise
$h(u,v)$	-	spatial distribution of writing beam ("impulse response) of recorder)
$m(u,v)$	-	"impulse response" of film
$n_g(u,v)$	-	grain noise
$T(u,v)$	-	recorded visibility data (film transmittance)

B. Processor Lens System

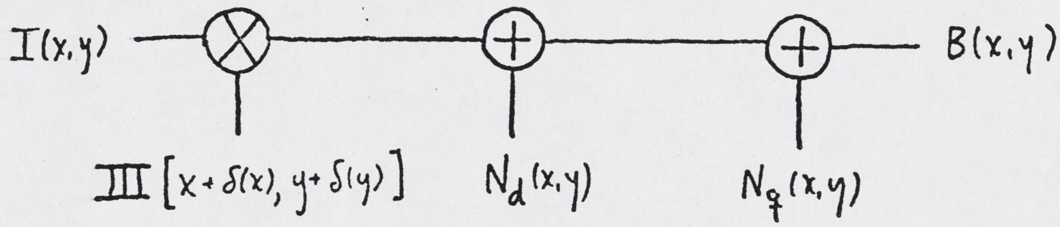


$T(u,v)$ -	recorded visibility data
$w(u,v)$ -	path length error function representing Fourier transform channel aberrations
$N_s(x,y)$ -	scattering noise
$L(x,y)$ -	path length error function representing reference beam channel aberrations.
$\psi(x,y)$ -	path length error function representing deviation of phase shifter from spatially uniform value of π

$k = 2\pi/\lambda$

Note: the output intensity $I(x,y)$ contains a term proportional to the desired brightness image $B(x,y)$. See p. 3, Bulabois (1), for verification.

C. Imaging Detector



$N_d(x,y)$ - detector noise

III(x,y) - two-dimensional sampling function representing finite detector resolution

$\delta(x)$, $\delta(y)$ - detector array positioning errors in the x, y directions respectively

$N_q(x,y)$ - quantization noise in A/D conversion

Note: It has been assumed that the effects of non-uniform spatial responsivity can be calibrated out once the system is constructed, and therefore are not included in this model.

CHARACTERIZATION OF MODEL ELEMENTS

For the purposes of later analysis, we will describe the model elements in terms of their behavior in the spatial frequency domain. The spatial domain is defined by the orthogonal coordinate system (u,v), while the frequency domain is defined by the orthogonal system (x,y). We denote the spatial frequency behavior of a model element by a capital letter, and its spatial behavior by a lowercase letter, i.e. $n_e(u,v) \Leftrightarrow N_e(x,y)$.

Since the noise elements are stochastic processes, their spatial frequency representations are also stochastic. Hence, we can only refer to the "generalized transform" of such a process (2). Because of the linearity of the transform, we can calculate the statistics of the frequency domain process, given the spatial domain process statistics.

A. Optical Recorder

Electronics noise: Usually regarded as wideband thermal noise, having Gaussian amplitude distribution $n_e(u,v) \iff N_e(x,y)$

Recorder impulse response: Ideally, the recorder writing beam could be expressed as a delta function. In practice, however, the writing beam has some spatial distribution of energy which we designate as h(u,v) - the "impulse response" of the recorder. In addition, there may be errors in the positioning of the beam. Including position errors, the impulse response can then be written as $h\left[u + \varepsilon(u),\ v + \varepsilon(v)\right]$, where $\varepsilon(u)$, $\varepsilon(v)$ are position errors in the u,v directions respectively. The linear system theoretic view of the writing beam distribution as an impulse response function has been justified elsewhere (4). The effect of beam positioning errors will be dealt with further in the analysis section of this memo. Note that

$$H'(x,y) = F\left\{h\left[u + \varepsilon(u),\ v + \varepsilon(v)\right]\right\} \tag{1}$$

Film impulse response: the spatial frequency response of film is determined empirically. However, the response curve is fairly well represented by a Gaussian lowpass characteristic, i.e.

$$|M(x,y)|^2 = \exp - \left[\frac{(x^2 + y^2)}{\beta^2} \right] \qquad (2)$$

where the parameter β determines the bandwidth of the film. The film coherent transfer function, $M(x,y)$ is usually complex valued:

$$M(x,y) = |M(x,y)| \exp \left[-j\Omega(f) \right] \qquad (3)$$

In general, $\Omega(f)$ must be determined experimentally.

Film grain noise: There are several models for film grain noise described in the literature (5). The so-called "checkerboard" model will be used here, primarily because of its simplicity. While it is but an approximation to real grain noise, its use provides valuable insight to the problem of selecting optimum exposure conditions to achieve good signal to noise ratio in reconstructed images(6).

Using the "checkerboard" model, grain noise is characterized by an autocorrelation function

$$\phi(u,v) = \overline{T}(1 - \overline{T}) \left(1 - \frac{|u|}{\ell}\right)\left(1 - \frac{|v|}{\ell}\right), \quad \begin{matrix} |u| \leq \ell \\ |v| \leq \ell \end{matrix}$$

$$= 0 \qquad \text{elsewhere} \tag{4}$$

with corresponding Wiener spectrum

$$\Phi(x,y) = \overline{T}(1 - \overline{T})\, \ell^2 \operatorname{sinc}^2 (x\ell) \operatorname{sinc}^2 (y\ell) \tag{5}$$

where

$\operatorname{sinc}(x) = \dfrac{\sin(\pi x)}{\pi x}$

$\overline{T}$ = average film transmittance

ℓ^2 = area of single grain

For the purpose of modeling, the actual noise process is referred to as $n_g(u,v)$ <=> $N_g(x,y)$ where $n_g(u,v)$ is approximated by a binomially distributed random process (5). In order that grain noise may simply be treated as additive noise, it is necessary to assume a low modulation level recorded signal. Unfortunately, this does not yield the best SNR. Under conditions of optimum SNR, it is found that the signal modulates the noise multiplicatively (7). This situation will not be treated here.

B. Processor Lens System

Fourier transform channel aberration: The problem of characterizing the effects of optical element aberrations on the spatial frequency response of imaging systems has not been treated in general, but only for simple, specific aberrations (3). It is extremely difficult to obtain closed form expressions representing these effects, and most probably will require numerical methods for obtaining results. For the time being, we simply denote the transform of the phase error function $\exp\left[jkw(u,v)\right]$ by $W(x,y)$.

Scattering Noise: $N_s(x,y)$ is assumed to be the transform of a Gaussian distributed amplitude, uniformly distributed phase, random process $n_s(u,v)$, since the noise arises from a large number of small scattering surfaces.

Reference beam channel aberrations: Remarks concerning the transform channel apply here as well. The phase error function for the reference channel is denoted by $\exp\left[jkL(x,y)\right]$.

Non-uniformity in phase shifter: this is essentially another aberration in the reference channel, and will simply be designated $\psi(x,y)$.

C. Imaging Detectors

Detector noise: There are actually two sources of noise in the detector:

(1) Thermal noise (or dark current)

(2) shot noise

However, at room temperature, thermal noise dominates shot noise in semi-conductor detector arrays (7). Thus, we can consider the noise $N_d(x,y)$ to be a wideband, Gaussian amplitude distributed random process.

Finite Detector Resolution and Array Positioning errors: The effect of finite detector resolution can be represented by a two-dimensional sampling function III (x,y). Errors in positioning the detector array require us to consider the output as being sampled with non-uniform spacing, i.e. with sampling function III $\left[x + \delta(x),\ y + \delta(y)\right]$, where $\delta(x)$, $\delta(y)$ are position error terms in the x,y directions respectively.

Quantization noise: Since storage of the output map is most easily accomplished in a digital fashion, it will be necessary to perform A/D conversion on the detector array output. This gives rise to quantization noise, a phenomenon which is exhaustively discussed in the digital signal processing literature. $N_q(x,y)$ is assumed to be a stationary white-noise process, uniformly distributed in the quantization step interval, and uncorrelated with the signal. Justification of these assumptions may be found elsewhere (8,9).

This completes the general description of the elements of the model.

System Analysis Using Proposed Model

Based on the model, we can easily write a very general expression relating the output brightness map $B(x,y)$ to the ideal map $\hat{B}(x,y)$.

Let $V(u,v) = A(u,v)\, e^{j\phi(u,v)}$ represent the complex visibility data. If we record this data in a quasi-sinusoidal format, yielding a new real-valued function

$$T(u,v) = A_o + kA(u,v)\cos\left[\omega_c u + \phi(u,v)\right] \qquad (6)$$

then the ideal Fourier transform is given by

$$F\left\{T\ (u,v)\right\} = A_o\delta(x,y) + k\left[\hat{B}(x - \frac{\omega_c}{2\pi}, y) + \hat{B}^*(x + \frac{\omega_c}{2\pi}, y)\right] \qquad (7)$$

where $\hat{B}(x,y) = F\left\{V(u,v)\right\}$ is the desired ideal brightness map. It is assumed that ω_c, the spatial carrier frequency, is chosen high enough so that $B(x - \omega_c/2\pi,y)$ and $B^*(x + \omega_c/2\pi,y)$ do not overlap.

The term $A_o\delta(x,y)$ is of minor concern, since it appears on the outside edge of the displaced version of the map, and by choosing ω_c properly, it can be made to fall entirely outside the map region of interest. Thus, we are interested in following only the term $\hat{B}(x - \omega_c/2\pi, y)$ as it propagates through the system. The equations involved in the analysis become rather lengthy if all terms are carried through, so some effort has been made to segment the problem and use compacted notation.

If we define the intermediate signal S(x,y) as the output of the processor before combination with the reference beam, and prior to the addition of scattering noise, we can write a very general expression relating S(x,y) to $\hat{B}(x - \omega_c/2\pi, y)$:

$$S(x,y) = \left(\left[k\hat{B}(x - \omega_c/2\pi, y) + N_e(x,y)\right] \cdot H'(x,y) \cdot M(x,y) + N_g(x,y)\right) * W(x,y) \qquad (8)$$

where * denotes convolution.

It is appropriate to make a few general observations at this stage of the analysis:

- H'(x,y) and M(x,y) tend to distort the outlying sections of the brightness map more severely than interior points, since both transfer functions are of a lowpass nature.
- convolution of the map with W(x,y) tends to "smooth" the map, reducing the resolution of the system.

We now proceed to find the output intensity I(x,y) after addition of the reference wave to the signal S(x,y). For compactness of notation, explicit designation of functional dependence on (x,y) has been dropped; it is to be understood that I(x,y) <=> I. The output intensity is given by

$$I = \left| (S + N_s) \cdot e^{j\theta} + R_o e^{j(kL + \theta_o)} \right|^2 \qquad (9)$$

where the factors $e^{j\theta}$ and $e^{j\theta_o}$ represent the spherical phase behavior of the illuminating waves in the transform and reference channels, respectively.

Expanding this expression,

$$I = R_o^2 + \left| (S + N_s) \cdot e^{j\theta} \right|^2 + 2R_o \cdot \mathrm{Real} \left[e^{-j(kL + \theta_o - \theta)} \cdot (S + N_s) \right] \qquad (10)$$

The term of interest to us, containing the brightness map, is

$$2R_o \cdot \mathrm{Real} \left[e^{-j(kL + \theta_o - \theta)} \cdot (S + N_s) \right] =$$

$$2 \cdot \mathrm{Real} \Big\{ \big(\big(\big[(kR_o \hat{B} + R_o N_e) \cdot H' \cdot M + R_o N_g \big] * W + R_o N_s \big) \cdot e^{-j(kL + \theta_o - \theta)} \Big\} \qquad (11)$$

A significant problem is that of separating this desired signal from the others present in the output intensity. The extraneous terms $\left(R_o^2, |(S + N_s) \cdot e^{j\theta}|^2\right)$ constitute noise which is superposed on the signal, and which tends to constrict the dynamic range of the system. If the noises are slowly varying in time, then the subtraction scheme proposed by Bulabois should be suitable for separating out the desired information. For more rapidly time-varying noise, several repetitions of the Bulabois measurements may be necessary to effectively "integrate" out the noise. Still another approach is to temporally modulate the phase of the reference wave, electronically filter the output to remove the slowly time-varying and non-varying noise terms, and synchronously demodulate to obtain the desired information. Which of these methods to employ has yet to be decided. It should be remembered that a primary source of time-varying noise is in the detector itself (thermal noise), and is essentially wideband. It seems reasonable that some improvement in SNR would result through use of the temporal modulation scheme, since the power in the time-varying (and non-varying) noises would be reduced through filtering. The model may be of assistance in determining which approach is most desirable.

ANALYSIS OF BEAM POSITION ERROR EFFECTS

If position errors $\varepsilon(u)$, $\varepsilon(v)$ in the input writing beam are random processes, then the exact description of the filtered signal cannot be obtained. Instead, we must treat the output map as a random signal contaminated by random noise. An analysis of recorder motion errors for matched filter processing has been performed by Harger (10). A similar approach will be employed here to determine the effect of position errors on the accuracy of the sky brightness map. For convenience, we restrict our attention to the one dimensional problem. As was shown by Harger, the recorded signal $\tilde{V}(u)$ can be written in the form

$$\tilde{V}(u) = \int dx\, e^{j2\pi xu}\, H(x) \cdot \int d\beta\, V(\beta) e^{-j2\pi x\left(\beta - \varepsilon(\beta)\right)} \qquad (12)$$

where $H(x)$ is Fourier transform of $h(u)$
$V(u)$ is ideal signal to be recorded
$\varepsilon(u)$ is the beam positioning error
$h(u)$ is the spatial distribution of the writing beam

This expression is entirely equivalent to

$$\tilde{V}(u) = \int dx\, V(x)\, h\left(u - x - \varepsilon(u)\right)$$

$$= V(u) * h\left[u - \varepsilon(u)\right] \qquad (13)$$

which coincides with the result obtained by Cindrich (3), and which justifies our intuitive expectation that the recorded signal is the convolution of the ideal signal with the recorder impulse response.

If we take the Fourier transform of the first expression for $\tilde{V}(u)$, we obtain

$$\tilde{B}(x) = H(x) \int d\beta\, V(\beta) e^{-j2\pi x\left(\beta - \varepsilon(\beta)\right)} \qquad (14)$$

If we assume that $\varepsilon(u)$ is a random process, then it is immediately apparent that $\tilde{B}(x)$ is also random. We can view the above equation as representing a cascade of filtering operations on $V(u)$:

- a spatial-invariant filter H(x)
- followed by a filter with response determined by the spatial variation of the stochastic process ε(u).

To obtain an idea of the effect of a random ε (u) on the resultant map $\tilde{B}(x)$, we take the expected value of both sides of (14):

$$\overline{\tilde{B}}(x) = H(x)\int d\beta\, V(\beta)e^{-j2\pi x\beta} \cdot \overline{e^{j2\pi x\varepsilon(\beta)}} \qquad (15)$$

Where overbar denotes expectation.

If we assume ε(β) is ergodic, then the expectation over β is the same as the expectation of a single sample function ε(β,ζ) taken from the process ε(β).

Thus, we can bring the term $\overline{e^{j2\pi x\varepsilon(\beta)}}$ outside the integral, since it is no longer a function of β. We have

$$\overline{\tilde{B}}(x) = H(x) \cdot \overline{e^{j2\pi x\varepsilon(\beta)}} \cdot \int d\beta\, V(\beta)e^{-j2\pi x\beta}$$

$$= H(x) \cdot \overline{e^{j2\pi x\varepsilon(\beta)}} \cdot B(x) \qquad (16)$$

We recognize $\overline{e^{j2\pi x\varepsilon(\beta)}}$ as the characteristic function of the process ε(β).

As a special case, assume $\varepsilon(\beta)$ is uniformly distributed on the interval $(-\alpha/2, \alpha/2)$, i.e. $\varepsilon(\beta)$ has probability density function

$$f_\varepsilon(a) = \frac{1}{\alpha} \qquad -\alpha/2 \leq a \leq \alpha/2$$
$$= 0 \qquad \text{elsewhere} \tag{17}$$

It is readily shown that the characteristic function for this case is

$$\overline{e^{j2\pi x\varepsilon(\beta)}} = \text{sinc}\ (x\alpha). \tag{18}$$

Thus,

$$\overline{\tilde{B}}(x) = H(x) \cdot \text{sinc}\ (x\alpha) \cdot B(x) \tag{19}$$

We see that, on the average, the effect of the random positioning errors is to attenuate the high spatial frequency components of the visibility data, or equivalently, to distort the outlying sections of the brightness map.
Note that

$$H'(x) = H(x)\ \text{sinc}\ (x\alpha) \tag{20}$$

for this case.

Extending our consideration to the two-dimensional case, we note that if $\varepsilon(u)$ and $\varepsilon(v)$ are uncorrelated random processes, we can write

$$H'(x,y) = H(x,y)\ \mathrm{sinc}\ (x\alpha)\ \mathrm{sinc}\ (y\alpha) \tag{21}$$

for the special case previously described ($\varepsilon(u)$,$\varepsilon(v)$ are identically distributed). We can consider the "average" filter $H'(x,y)$ as being representative of the effects of positioning errors on the output brightness map.

General Comments

It should be possible to derive an expression for the mean-squared error in the output brightness map, in terms of the system model parameters. However, it appears that the arithmetic may be very tedious. A better approach would be computer simulation. This would enable one to determine the resolution and dynamic range of the system as a function of the film parameters, noise statistics, etc. While only a first order approximation, the model may help to pinpoint problem areas in the system and provide a basis for estimating system performance.

References

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(2) A. Papoulis, "Probability, Random Variables, and Stochastic Processes", McGraw-Hill (1965).

(3) Born and Wolf, "Principles of Optics", 2nd edition Pergamon Press (1964)

(4) I. Cindrich, "Scanning Beam Model with Scan Position Errors." ERIM Internal Memo (VLA Optical Processor File) July 19, 1976.

(5) E.L. O'Neill, "Introduction to Statistical Optics", Addison-Wesley (1963).

(6) J.W. Goodman, "Film-grain Noise in Wavefront Reconstruction Imaging", J. Opt. Sci. Am. Vol. 57, No. 4 April 1967.

(7) K. Preston, Jr., "Coherent Optical Computers," McGraw-Hill (1972).

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(10) R.O. Harger, "An Analysis of Recorder Motion Errors in Optical Processing", Applied Optics, Vol. 4, No. 4 April 1965.

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