2 September 1976

MEMORANDUM TO: VLA Optical Processor File

FROM: James R. Fienup J. F.

SUBJECT: Phase Modulation Encoding Method

In this memo I show that a phase-modulation encoding method is limited to low diffraction efficiency in order to reduce the intrinsic phase nonlinearity. Considering its other disadvantages, this makes the phase-modulation method a poor candidate for the VLA optical processor.

Suppose that we produce a pure-phase transparency to encode the desired wavefront,

$$F(u,v) = 2|V(u,v)| \cos [\omega_o u + \phi(u,v)]$$

$$= V(u,v)e^{j\omega_o u} + V^* (u,v)e^{-j\omega_o u}.$$

The desired term, $V(u,v)e^{j\omega_o u}$, when Fourier transformed, produces $B(x,y)$ shifted by $\omega_o \lambda f/2\pi$. The transmittance of such a transparency, assuming a perfectly linear (or else perfectly compensated) material characteristic curve is

$$H(u,v) = e^{j\alpha F(u,v)}$$

This relationship between F and H is intrinsically nonlinear and introduces spurious terms into the reconstruction that overlap and degrade the desired term, $V(u,v)e^{j\omega_o u}$. Expanding, we have

$$H(u,v) = e^{j\alpha F}$$

$$= 1 + j\alpha F + \frac{(j\alpha)^2}{2} F^2 + \frac{(j\alpha)^3}{3!} F^3 + \frac{(j\alpha)^4}{4!} F^4 + \ldots$$

$$= 1 + j\alpha V e^{j\omega_o u} + j\alpha V^* e^{-j\omega_o u}$$

$$+ \frac{(j\alpha)^2}{2} \left[V^2 e^{j2\omega_o u} + 2|V|^2 + V^{*2} e^{-j2\omega_o u} \right]$$

$$+ \frac{(j\alpha)^3}{6} \left[V^3 e^{j3\omega_o u} + 3V|V|^2 e^{j\omega_o u} + 3V^*|V|^2 e^{-j\omega_o u} \right.$$

$$\left. + V^{*3} e^{-j3\omega_o u} \right] + \ldots$$

where the spatial coordinates (u,v) are understood.

If the width of the image due to V(u,v) is W, then the width of a term due to $[V(u,v)]^n$ is nW. All terms of the form $V^n e^{jn\omega_o u}$ can be made to not overlap the desired term, $Ve^{j\omega_o u}$ if the carrier $\omega_o u$ is made great enough to move the desired image by a distance 1.5W from the optical axis (requiring four times the bandwidth as required by V(u,v) above). In particular, we must avoid overlap with the term $|V|^2$ which results in an autocorrelation of

B(x,y) and with the term V^2 which results in a convolution of B(x,y) with itself.

There remain, however, terms of the form $\frac{(j\alpha)^n}{(n-1)!} \cdot V|V|^{n-1}e^{j\omega_o u}$, n = 3,5,7,... that cannot be spatially separated from the desired image. The largest such "intermodulation" term is for n=3: $-\frac{j\alpha^3}{2} V|V|^2 e^{j\omega_o u}$. The sum of this term with the desired term is $j\alpha V\left[1 - \frac{\alpha^2}{2}|V|^2\right]$, which yields the image term equal to $\alpha B(x,y) - \frac{\alpha^3}{2} B(x,y) * [B(x,y) \star B(x,y)]$ where $*$ and $\star$ are the correlation and autocorrelation operations, respectively.

The total amount of energy going into the desired term is given by

$$\eta_1 = \iint_{-\infty}^{\infty} |\alpha V(u,v)|^2 \, dudv \equiv \alpha^2 \overline{|V|^2}$$

and the energy going into the n=3 intermodulation term above is given by

$$\eta_3 = 1/4 \iint_{-\infty}^{\infty} |\alpha V(u,v)|^6 \, dudv \equiv \frac{\alpha^6}{4} \overline{|V|^6}$$

For a constant $|V(u,v)|$, $\overline{|V|^6} = \left(\overline{|V|^2}\right)^3$. Using this as an approximation we have

$$\eta_3/\eta_1 = \frac{\alpha^4}{4}\left(\overline{|V|^2}\right)^2$$

or

$$\eta_3/\eta_1 = \eta_1^{\,2}/4 \ .$$

If B(x,y) were to be a uniform extended object, then we would need $\eta_3/\eta_1 < 10^{-4}$ in order to satisfy the 1% criterion; consequently, the diffraction efficiency of the desired image would be limited to $\eta_1 < 0.02$. Then the chief advantage of a phase recording, its high diffraction efficiency, cannot be taken advantage of. Considering another extreme example, suppose that $B(x) = \alpha[\delta(x-x_2) + A\,\delta(x-x_2)]$ where $A \leq 1$. (Then $\eta_1 = \alpha^2(1+A)^2$.)
The intermodulation noise is given by

$$\frac{\alpha^3}{2} B(x) * [B(x) \star B(x)] = \frac{\alpha^3}{3}[(1+2A^2)\delta(x-x_1)+(2A+A^3)\delta(x-x_2)$$

$$+A^2\delta(x-2x_2+x_1) + A^2\delta(x+x_2-2x_1)] .$$

Taking the case of A=1, for which this term is maximized, it has the maximum delta-function amplitude of α^3 at points x_1 and x_2. Then the ratio of the amplitudes of the intermodulation term to the desired term is $\alpha^3/\alpha = \alpha^2$. Keeping this less than 0.01 requires that $\eta_1 < 0.04$. For this case as well, there is no advantage to using a phase-only transparency as far as diffraction efficiency is concerned.

Thus, the use of a phase modulation transparency would be advantageous only if the film noise from amplitude modulation materials were the limiting factor and if the phase modulation material were to solve this problem. A disadvantage of using a phase modulation material is the presence of the undiffracted wavefront that is transmitted through the areas not covered by tracks (the same as with a negative material). Another disadvantage is the requirement of using a higher carrier frequency in order to avoid the autocorrelation term centered at the optical axis and the cross-correlation term centered at the second harmonic. It would also be required to use a phase material whose phase modulation as a function of spatial frequency is constant over the range of spatial frequencies used. (Although compensation for a non-constant response is possible, such compensation is totally impractical because it would involve the computation of a very large Fourier transform.)

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