8 September 1976

MEMORANDUM TO: VLA File

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SUBJECT: Sampling Considerations

In this memo we consider some aspects of sampling with finite sized apertures.

Let V(u,v) be the visibility function and B(x,y) its FT, namely, the brightness function. The visibility function is assumed hermitian and hence the brightness is real.

The input signal is

$$s(u,v) = V(u,v) + R\delta(u - u_r)e^{i\phi} \tag{1}$$

where R, ϕ, and u_r are the reference wave constants. The output signal S(x,y) is the FT of s(u,v) and is given by

$$S = B + Re^{i\phi}e^{-2\pi i u_r x} \tag{2}$$

The local intensity is

$$I(x,y) = |S|^2$$

$$= |B|^2 + |R|^2 + 2BR \cos (2\pi u_r x + \phi) \tag{3}$$

A two dimensional array of identical detectors with periodic spacing d_x and d_y detect the signal

$$E(x,y) = [I(x,y) * A(x,y)] \cdot \text{III}\left(\frac{x}{d_x}\right) \text{III}\left(\frac{y}{d_y}\right) \qquad (4)$$

where A(x,y) is the shape of an individual detector aperture,

$$\text{III}(x) = \sum_{n=-\infty}^{\infty} \delta(x - n) \qquad (5)$$

is the comb function, and * represents a two dimensional convolution.

The difference in intensity between E(x,y) with phase ϕ_2 and then with phase ϕ_1 is given by

$$\Delta E = (A * \Delta I)\text{III}\left(\frac{x}{d_x}\right) \text{III}\left(\frac{y}{d_y}\right) \qquad (6)$$

where

$$\Delta I = 2BR\,[\cos(2\pi u_r x + \phi_2) - \cos(2\pi u_r x + \phi_1)]$$

$$= 4BR \sin(\alpha) \sin(2\pi u_r x + \beta) \qquad (7)$$

and

$$\alpha = \frac{\phi_2 - \phi_1}{2}, \quad \beta = \frac{\phi_1 + \phi_2}{2} \qquad (8)$$

Thus, the measured signal ΔE is a sampled version of $A * \Delta I$. The inverse FT of ΔE is the effective input signal, i.e., the input signal for which a simple FT would give that same output as (6), namely

$$\Delta e = (a \cdot \Delta i) * d_y d_x \text{III}(d_x x)\ \text{III}(d_y y) \qquad (9)$$

where $a(u,v)$ is the FT of $A(x,y)$ and

$$\Delta i = -2iR \sin(\alpha)$$

$$[V(u-u_r,v)e^{i\beta} - V(u+u_r,v)e^{-i\beta}] \qquad (10)$$

is the FT of ΔI. Thus, the effective input signal is a replicated version of the function $a\Delta i$. If $V(u,v)$ is limited in extent to $(\pm U_o/2, \pm V_o/2)$ and if (see Figure 1)

$$\frac{1}{d_y} > V_o, \qquad (11a)$$

$$\frac{1}{d_x} > U_o + 2u_r \qquad (11b)$$

and

$$u_r > \frac{U_o}{2} \text{ or } u_r = 0 \qquad (11c)$$

then the signal $a\Delta i$ can be recovered exactly. In the limit of a small aperture, i.e., $A \to \delta(x,y)$ and hence $a(u,v) \to 1$, then Δi can be found exactly and sampled values of B are directly found from (6) and (7) as

$$B(m', n') = \frac{\Delta E(m',n')}{4R \sin \alpha \sin (2\pi m' + \beta)} \qquad (12)$$

where

$$m' = m \frac{u_r}{d_x} \qquad m,n = 0, \pm 1, \pm 2, \ldots$$

$$n' = n \frac{u_r}{d_x}$$

as long as R, α, β, and u_r are known accurately. Obviously choosing $\alpha = \beta = \pi/2$ (i.e., $\phi_1 = 0$, $\phi_2 = \pi$), $d_x = u_r$, and $R = 1/4$ gives B directly, namely from (12)

$$B(m,n) = \Delta E(m,n) \qquad (13)$$

It is clear that $\sin \alpha$, $\sin \beta$, R need to be known to much better than 1% if B is to be known to 1%. [Note: arccos (.99) = $\pm 8.1^o$.]

Let us now specifically consider the detector aperture effects. Towards this end let us simplify the equations by letting $\alpha = \beta = \pi/2$ and $R = 1/4$. Then (6) and (9) become

$$\Delta E = A * [B \cos (2\pi u_r x)] \tag{14}$$

and

$$\Delta e = \frac{1}{2} a \cdot [V(u - u_r, v) + V(u + u_r, v)] \tag{15}$$

where the sampling and replication effects have been dropped. It is clear that the finite size aperture degrades the output signal by its convolution with the desired signal. A point star corresponding to $B = \delta(u,v)$ gives a spread function $A(u,v)$, or equivalently, the frequency response of the system is given by $a(u,v)$. We also see that if the input signal were properly tapered, that is, if $V(u,v)$ is replaced by

$$\frac{V(u,v)}{a(u + u_r, v)}$$

then (15) becomes

$$\Delta e = \frac{1}{2} V(u - u_r, v) + \frac{1}{2} \frac{a(u,v)}{a(u + 2u_r, v)} V(u + u_r, v) \tag{16}$$

which if filtered to bandpass the first term would give the desired result. However, this is probably not practical due to the computational effort involved except for the special case where $u_r = 0$. In this case (16) becomes

$$\Delta e = V(u,v) \tag{17}$$

and hence

$$\Delta E = B \tag{18}$$

which is exactly the desired result obtained without any additional processing. The above results indicate a strong reason to place the reference wave source at the center of the input signal, i.e., to let $u_r = 0$. However, if A(x,y) is sufficiently small then a(u,v) is nearly uniform and the input taper is not necessary. That is, if W_x and W_y are the characteristic aperture widths in the x and y directions respectively, then

$$W_x << \frac{1}{u_r} < \frac{1}{U_o}$$
$$W_y << \frac{1}{V_o} \tag{19}$$

are the conditions to be satsified. However, how well the conditions (19) should be satisfied to satisfy the 1% criteria should be looked at more carefully. It is likely that A(x,y) would need to be measured experimentally in order to obtain sufficient information to analyze the problem.

In general the aperture can be modified by overlaying some mask over the array. However, such an arrangement would reduce the energy available to the detectors.

In some of the detector arrays under consideration the aperture size of the individual detectors is nearly equal to their interspacing (i.e., $W_x = d_x$). This implies that the output must be considerably over-sampled in order to satisfy conditions (19).

With a one-dimensional array along the y-direction, where moving the entire array gives information in the other (x) dimension, a single slit can be easily placed along the long dimension. Notice that the array and slit are parallel to the carrier fringes. The slit makes $W_x \ll W_y$. This minimizes the oversampling requirements along the reference offset direction x and light energy is lost only along one dimension.

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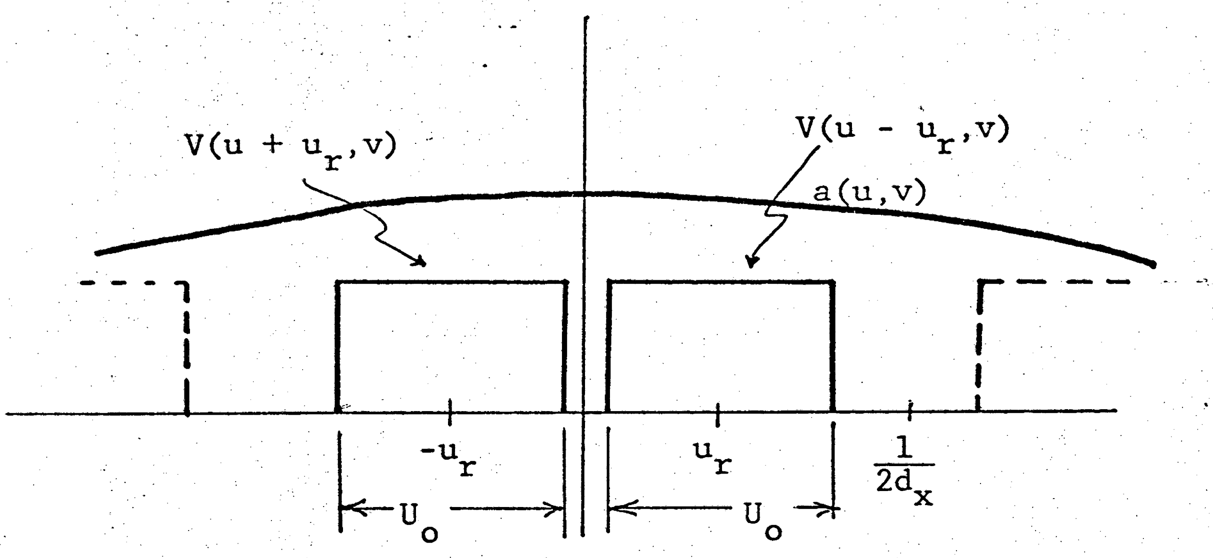

Figure 1