VLA SCIENTIFIC REPORT NO. 108

June 13, 1973

COMMENTS ON VLA SCIENTIFIC REPORT 106

C. M. Wade

Abstract: It is shown that the shortcomings of the VLA as a mapping instrument, as discussed in Report 106, are due largely to the definition of the mapping variables $\underline{x}$ and $\underline{y}$. A suitable alternative expression is derived, which eliminates part of the difficulties treated in Report 106. The approach used here ignores the effects of the curvature of the celestial sphere; it is shown that, for full-bandwidth continuum observations with the VLA, the phase errors arising from this neglect never exceed about 20°.

In VLA Scientific Report 106, Bob Hjellming drew attention to certain "higher order phase terms" which would have a quite destructive effect on the size of the usable field of view of the VLA. We show below that these terms are the result of (a) an inappropriate definition of the map coordinates $\underline{x}$ and $\underline{y}$ in terms of celestial coordinates, and (b) neglect of the curvature of the celestial sphere. The effects of the latter are relatively minor for continuum mapping with the VLA at the full system bandwidth.

It is sufficient to restrict the discussion to fringe phase as a function of source position relative to center of the field to be mapped. All terminology not otherwise defined follows the notation and conventions which have become standard in the VLA and interferometer groups.

The Basic Phase Equation

Let the unit vector $\vec{s}_o = \vec{s}_o\,(H_o, \delta_o)$ denote the center of the field, and assume that the baseline errors $\Delta\vec{B} = 0$. Then the phase of the fringes due to a source (or element of an extended brightness distribution) in the direction $\vec{s}$ is

$$\Phi = -\vec{B}\cdot(\vec{s} - \vec{s}_o) \tag{1}$$

in turns (see Ap. J. 162, 381, 1970).

In the (u,v,D) coordinate system, the vectors are:

$$\vec{B} = \begin{bmatrix} u \\ v \\ D \end{bmatrix} \equiv \begin{bmatrix} B_u \\ B_v \\ B_D \end{bmatrix} = \underset{\sim}{T} \begin{bmatrix} B_x \\ B_y \\ B_z \end{bmatrix} ,$$

$$\vec{s} = \begin{bmatrix} s_u \\ s_v \\ s_D \end{bmatrix} = \underset{\sim}{T} \begin{bmatrix} \cos\delta \cos H \\ \cos\delta \sin H \\ \sin\delta \end{bmatrix},$$

$$\vec{s}_o = \begin{bmatrix} s_{u,o} \\ s_{v,o} \\ s_{D,o} \end{bmatrix} = \underset{\sim}{T} \begin{bmatrix} \cos\delta_o \cos H_o \\ \cos\delta_o \sin H_o \\ \sin\delta_o \end{bmatrix} ,$$

where

$$\underset{\sim}{T} = \begin{bmatrix} \sin H_o & -\cos H_o & 0 \\ -\sin\delta_o \cos H_o & -\sin\delta_o \sin H_o & \cos\delta_o \\ \cos\delta_o \cos H_o & \cos\delta_o \sin H_o & \sin\delta_o \end{bmatrix} .$$

Then

$$\begin{bmatrix} s_u \\ s_v \\ s_D \end{bmatrix} = \begin{bmatrix} \cos\delta\,(\sin H_o \cos H - \cos H_o \sin H) \\ -\sin\delta_o \cos\delta\,(\cos H_o \cos H + \sin H_o \sin H) + \cos\delta_o \sin\delta \\ \cos\delta_o \cos\delta\,(\cos H_o \cos H + \sin H_o \sin H) + \sin\delta_o \sin\delta \end{bmatrix} .$$

Since $\Delta\alpha = H_o - H$, this can be written

$$\begin{bmatrix} s_u \\ s_v \\ s_D \end{bmatrix} = \begin{bmatrix} \cos\delta \sin\Delta\alpha \\ -\sin\delta_o \cos\delta \cos\Delta\alpha + \cos\delta_o \sin\delta \\ \cos\delta_o \cos\delta \cos\Delta\alpha + \sin\delta_o \sin\delta \end{bmatrix} .$$

Also,

$$\begin{bmatrix} s_{u,o} \\ s_{v,o} \\ s_{D,o} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} .$$

The vector position of the source with respect to the center of the field is then

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \equiv \vec{s} - \vec{s}_o = \begin{bmatrix} \cos\delta \sin\Delta\alpha \\ -\sin\delta_o \cos\delta \cos\Delta\alpha + \cos\delta_o \sin\delta \\ \cos\delta_o \cos\delta \cos\Delta\alpha + \sin\delta_o \sin\delta \quad -1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 \\ -\sin\delta_o & 0 & \cos\delta_o \\ \cos\delta_o & 0 & \sin\delta_o \end{bmatrix} \begin{bmatrix} \cos\delta \cos\Delta\alpha \\ \cos\delta \sin\Delta\alpha \\ \sin\delta \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad (2)$$

The fringe phase for a source at this location is therefore

$$\Phi = -ux - vy - Dz . \qquad (3)$$

This is a general result, derived without approximations. It differs from the corresponding expression in previous NRAO practice in two ways. First, $\underline{x}$ and $\underline{y}$ are defined differently; previously we have used definitions derived from a small angle approximation (see next paragraph). The old definition ignores the fact that the projection of a grid of points in (α,δ) onto the tangent plane can never form a truly rectangular grid. Second, eq. (3) has a z-term because the ensemble of position vectors s defines a unit sphere (not a plane surface) which is in contact with the tangent plane only at s_o.

Report 106 relies implicitly on the assumption that $\Delta\alpha \ll 1$ and $\delta - \delta_o = \Delta\delta \ll 1$. Under these assumptions, eq. (2) reduces to

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \Delta\alpha \cos\delta \\ \Delta\delta \\ 0 \end{bmatrix} .$$

This forces the (α,δ) grid to be rectangular on the tangent plane. It is the artificiality of this rectangular grid which causes the declination dependence of the large phase effects considered in Report 106. The remaining part is due to the neglect of the curvature of the unit sphere.

Mapping and Neglect of the Curvature Term

In order to be rigorously correct, one should include the z-term of eq. (3) in the Fourier transform relation. We shall not do so here, since z is always very small. Instead, we shall evaluate the effect of ignoring it. We assume that the map in terms of x and y is calculated in the usual way, but with x and y defined by eq. (2). The conversion to (α,δ) is then obtained by inverting eq. (2) under the assumption that $z = 0$:

$$\begin{bmatrix} \cos\delta\cos\Delta\alpha \\ \cos\delta\sin\Delta\alpha \\ \sin\delta \end{bmatrix} = \begin{bmatrix} 0 & -\sin\delta_o & \cos\delta_o \\ 1 & 0 & 0 \\ 0 & \cos\delta_o & \sin\delta_o \end{bmatrix} \begin{bmatrix} x \\ y \\ 1 \end{bmatrix} . \tag{4}$$

This gives an orthographic projection of the celestial sphere onto the tangent plane.

When we set $z = 0$, we introduce a phase error which is small near the center of the field, but which increases with the square of the distance from the center of the field. The error is

$$\Delta \Phi = -Dz .$$

Now

$$z = \vec{s} \cdot \vec{s}_o - 1 = \cos\theta - 1 ,$$

where θ is the angle between $\vec{s}$ and $\vec{s}_o$. Since θ is always small, we have

$$\Delta\Phi = \frac{1}{2}\theta^2 D .$$

For an approximate "worst case" discussion, we can take $D \sim L\lambda^{-1}$, where L is the length of an array arm and λ is the observing wavelength.

The admissible range of θ is controlled by either the primary beamwidth or the delay beamwidth, depending on wavelength, bandwidth, and array configuration:

(a) <u>θ limited by the delay beamwidth β_d</u>:

We have approximately, in radians,

$$\beta_d \simeq \frac{2c}{L\Delta\nu}$$

where <u>c</u> is the velocity of light and $\Delta\nu$ is the bandwidth (single-channel). We can take the maximum value of θ to be approximately $\frac{1}{2}\beta_d$, so

$$\Delta\Phi \lesssim \frac{c^2}{2\lambda L\ \Delta\nu^2} . \tag{5a}$$

Since $\Delta\nu \simeq 5 \times 10^7$ H_z for continuum observations with the VLA, we have

$$\boxed{\Delta\Phi \lesssim \frac{18}{L\lambda} \quad \text{(turns)}} \qquad (5b)$$

where λ and L are in meters.

(b) $\underline{\theta \text{ limited by the primary beamwidth } \beta_p}$:

In this case we take

$$\theta \;\leq\; \frac{1}{2}\,\beta_p \approx \frac{1}{2}\,\lambda d^{-1} \;,$$

where d = 25 m is the diameter of a single antenna. Then

$$\Delta\Phi \lesssim \frac{\lambda\ L}{8d^2} \qquad (6a)$$

or

$$\boxed{\Delta\Phi \lesssim \frac{\lambda\ L}{5000} \quad \text{(turns)}} \qquad (6b)$$

where λ and L are again in meters.

Eqs. (5b) and (6b) give the same value for $\Delta\Phi$ when $\lambda L = 300\ m^2$. If $\lambda L < 300\ m^2$, θ is set by the primary beam; when $\lambda L > 300\ m^2$, it is limited by the delay beam (for continuum observations with the full bandwidth, remember). The largest phase error occurs for $\lambda L = 300\ m^2$, when $\Delta\Phi \lesssim 0.06$ turns = 22°.

The table below shows how the maximum phase error varies with wavelength and arm length for the VLA. It is assumed that the observations are made in the continuum with the full bandwidth. The values in paraentheses are set by the delay beamwidth; the other values are controlled by the primary beamwidth.

MAXIMUM $\Delta\Phi$ ($\Delta\nu$ = 50 MHz)

L \ λ	20 cm	6.3 cm	2.0 cm	1.3 cm
21 km	(1°.5)	(4°.9)	(15°)	20°
5.78 km	(5°.6)	(18°)	8°.3	5°.4
1.59 km	(20°)	7°.2	2°.3	1°.5
0.44 km	6°.3	2°.0	0°.6	0°.4

None of these errors is of a magnitude which would seriously degrade the mapping capability of the instrument. Serious errors may well be encountered, however, when the bandwidth is reduced greatly to enlarge the field of view, or for spectral line observations. According to eqs. (5a) and (6a), the delay beam will control θ when $\lambda L > 2cd/\Delta\nu$. For $\Delta\nu$ in MHz and d = 25 m, this is $\lambda L > 15000/\Delta\nu$ (m^2). The corresponding "worst case" phase error is $\Delta\Phi = 3/\Delta\nu$ turns = $1080°/\Delta\nu$.

Conclusion

The "higher order non-linear phase terms" of Report 106 are largely artifacts of the way $\underline{x}$ and $\underline{y}$ were defined. In particular, the strong declination dependence of these terms vanishes when $\underline{x}$ and $\underline{y}$ are defined in the manner suggested in the present report. The portion which remains is due to neglect of the curvature of the celestial sphere.

The curvature term Dz in eq. (3) can safely be ignored only for continuum mapping with the full bandwidth of the system. Practical means for taking this term into account must be found for spectral line mapping and continuum mapping with reduced bandwidth. This might require the use of a three-dimensional Fourier transform. It is to be hoped, however, that some less heavy-handed approach would be adequate. The matter deserves careful study.