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THE EFFECT OF BANDWIDTH ON THE SYNTHESIZED BEAM

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I. INTRODUCTION

The purpose of this memorandum is to give, in some detail, the effect of bandwidth on the synthesized beam. Dick Thompson, in the VLA Electronics Memorandum #118, has laid the groundwork and shown the effects of bandwidth on peak response. Here we extend the results and show the effects of bandwidth on the beam shape and the resolution.

We assume throughout that the u,v plane is fully sampled, and that the beam is related to this fully sampled u,v plane by an analytic Fourier transform. Thus, effects of aliasing, due to the FFT, are not considered. These will be negligible in virtually all practical cases when compared to the beam.

It is obvious that the bandpass shape and u,v taper will strongly influence the beam shape. An important conclusion which derives from this work is that these differences become small when the quantities of interest are considered functions of a universal parameter.

II. THE EFFECT OF BANDWIDTH ON VISIBILITY MEASUREMENTS

Thompson has shown that the bandwidth effect multiplies the true visibilities of the source in question by a function $g(\tau)$, where

$\tau =$ differential time delay for source in question from the phase center.

If the source position is (x,y), relative to the phase center, measured in radians, and the baseline coordinates are (u,v), measured in wavelengths, then

$$\tau = \frac{ux+vy}{\nu_o} \tag{1}$$

where ν_o is the observing frequency.

The bandwidth loss function can be evaluated from knowledge of the bandpass shape of the interferometer pair in question: they are Fourier transform pairs.

$$g(\tau) \supset G(\nu') = A_1(\nu')\, A_2^*(\nu') \tag{2}$$

where $\nu' = \nu-\nu_o$ is the frequency offset from the bandpass center, and A_1, A_2 are the passband shapes of the antennas contributing to the interferometer pair in question.

Considerable simplification, and good approximation to reality, is gained in assuming all antennas have the same passband. If we further assume the antenna bandpasses are even functions about ν_o, then the loss function, $g(\tau)$ is real and even. In what follows, we assume this to be the case, so the effect on the u,v data is a simple loss of amplitude.

Because rotation by θ in the map plane corresponds to rotation by the same angle in the u,v plane, we can consider, without loss of generality, the source to be displaced solely along the x-axis. Then the bandwidth loss function, $g(\tau)$ is solely a function of u. Thus, if the true visibilities are $V(u,v)$, then the measured visibilities are:

$$V_m(u,v) = V(u,v) \cdot g(u)$$

so the synthesized beam becomes:

$$B_D(x,y) = B(x,y) ** D(x)\delta(y)$$

where B_D and B represent the distorted and undistorted beams, $D(x)$ is the Fourier transform over u of $g(u)$, and $\delta(y)$ is the Kronecker delta function. The double asterisk (**) denotes two-dimensional convolution. If $B(x,y)$ is separable in x and y:

$$B(x,y) = b(x)c(y),$$

then we immediately find that

$$B_D(x,y) = c(y) \cdot [b(x) * D(x)].$$

Thus, the distortion caused by the bandwidth acts on the beam in only one coordinate - that line joining the phase center to the position in question. The beam shape in the orthogonal direction is unaffected. Although this applies strictly to "separable" beams (which include a Guassian), practical beams are usually of a form which differ only slightly from ideal, separable ones.

In the next section, we calculate the distorting function, D(x), and convolve it with the undistorted beam to find the distorted beam.

III. CALCULATION OF BEAMSHAPE FOR SOME SPECIAL CASES

Consider a point source of unit flux located at position x_o.

$$I(x,y) = \delta(x-x_o,y) = \delta(x-x_o)\delta(y).$$

The visibility function is then, simply,

$$V(u) = e^{-2\pi i x_o u} .$$

The interferometer measures this at points described by $S(u,v)$, the spatial measuring function. The effect of the bandwidth is to multiply the visibility by the bandwidth loss function, $g(u)$. Finally, an applied taper, $T(u,v)$ is applied. Thus, the visibility data, prior to transformation, is:

$$S(u,v)\bullet T(u,v)\bullet V(u,v)\bullet g(u)$$

We recognize the expression $S\bullet T\bullet V$ as the synthesized beam, free of bandwidth effects. As noted in the last section, we consider only cases where the distortion acts in one dimension. Then, the distorted beam is

$$B_D(x) = B(x)*D(x)$$

where $D(x)$, the distorting function, is given by the Fourier transform over u of the bandwidth loss function, $g(\tau)$.

We now consider special, simplified cases.

<u>Class A</u>

<u>Square bandpass of full width $\Delta\nu$</u>.

The (normalized) bandpass is:

$$G(\nu') = \frac{1}{\Delta\nu} \sqcap \frac{\nu'}{\Delta\nu}$$

And, the loss function in the u,v plane becomes, from equation (2)

$$g(\tau) = \text{sinc } (\Delta\nu\tau) = \text{sinc } (\frac{\Delta\nu}{\nu_o} x_o u)$$

Finally, the transform over u gives the distorting function,

$$D(x) = \frac{\nu_o}{\Delta\nu x_o} \sqcap \ (\frac{\nu_o}{\Delta\nu} \frac{x}{x_o}) \qquad (3)$$

The distorted beam is then the convolution of the undistorted beam with the above expression.

Case 1

Full u,v coverage within a square of width $2u_o$, and no u,v taper. u_o is the "maximum baseline".

Then,

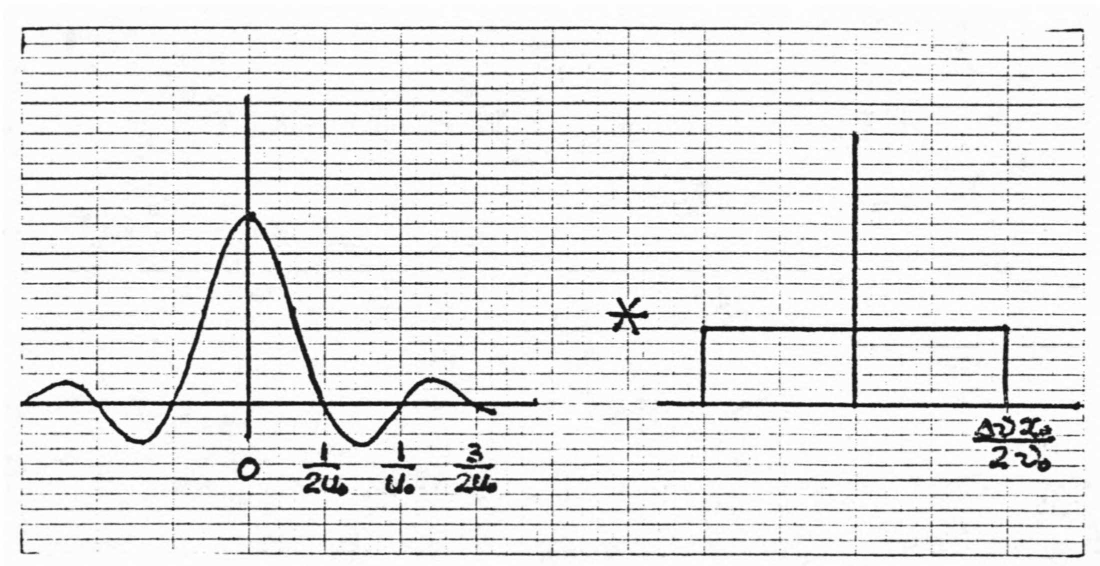
$$S(u) = \sqcap\left(\frac{u}{2u_o}\right)$$

$$T(u) = 1$$

Then the normalized beam is simply:

$$B(x) = \text{sinc}\,[2u_o(x-x_o)]\ \text{sinc}\,(2u_o y)$$

and we seek its convolution with (3), as pictured below.



At any given point, Δx, from the beam center, x_o, the value of this convolution is:

$$B_D(\Delta x, x_o) = \frac{\nu_o}{\Delta\nu x_o} \int_{\Delta x - \frac{\Delta\nu x_o}{2\nu_o}}^{\Delta x + \frac{\Delta\nu x_o}{2\nu_o}} \text{sinc } (2u_o x) dx$$

$$= \frac{\nu_o}{2\pi u_o \Delta\nu x_o} \left\{ Si[2\pi u_o \; (\Delta x + \frac{\Delta\nu x_o}{2\nu_o})] \; - Si[2\pi u_o \; (\Delta x + \frac{\Delta\nu x_o}{2\nu_o})] \right\} \quad (4)$$

where $Si(x) = \int_o^x \frac{\sin(t)}{t} dt$ is the sine integral function.

Now, the FWHP of the function sinc $(2u_o x)$ is

$$b = \frac{0.603}{u_o}$$

Defining $\eta = 2\pi u_o b = 3.79$

$$\alpha = \frac{\Delta x}{b} = \text{width parameter in undistorted beamwidths}$$

$$\beta = \frac{\Delta\nu}{\nu_o} \frac{x_o}{b} = \text{fractional bandwidth multiplied by the}$$

offset in beamwidths, we express (4) as

$$\boxed{B_D(\Delta x, x_o) = \frac{1}{\eta\beta} \{Si[\eta(\alpha + \frac{\beta}{2})] \; - Si[\eta(\alpha - \frac{\beta}{2}]\}} \quad (5)$$

At the beam center, $\Delta x = 0$, so $\alpha = 0$. Then,

$$B_D(0, x_o) = \frac{2}{\eta\beta} \; Si(\frac{\eta\beta}{2}) \quad (6)$$

This represents the 'peak response', or sensitivity. When the source offset, x_o, or the bandwidth, $\Delta\nu$, is zero, the beam is:

$$\text{sinc}\left(\frac{\eta\alpha}{\pi}\right) = \sin(\eta\alpha)/\eta\alpha.$$

We defer discussion of these functions until after the following example.

Case 2

Full u,v coverage to a radius q_o which is large compared to a Gaussian taper described by:

$$T(u) = \frac{\sqrt{\pi}b}{\gamma} \exp\left(\frac{-\pi^2 b^2 u^2}{\gamma^2}\right).$$

This has a half-power width at $u_{\frac{1}{2}} = \frac{\gamma^2}{2\pi b}$, where

$$\gamma = 2\sqrt{\ln 2} = 1.665$$

b = FWHP of synthesized beam.

The undistorted beam corresponding to this taper is

$$B(x) = \exp\left(-\frac{\gamma^2 x^2}{b^2}\right).$$

The calculation of the broadened beam proceeds as in the last case, and yields:

$$\boxed{B_D(\Delta x, x_o) = \frac{\sqrt{\pi}}{2\beta\gamma}\left\{\text{erf}\left[\gamma\left(\alpha + \frac{\beta}{2}\right)\right] - \text{erf}\left[\gamma\left(\alpha - \frac{\beta}{2}\right)\right]\right\}} \qquad (7)$$

where α and β are defined as before.

At the beam center, $\alpha = 0$, and

$$B_D(0,x_o) = \frac{\sqrt{\pi}}{\beta\gamma} \operatorname{erf}\left(\frac{\gamma\beta}{2}\right) \tag{8}$$

which is identical to Thompson's equation (8).

Similar (but more complicated) expressions can be generated for other combinations of taper and sampling.

<u>Class B</u>

A Gaussian bandpass of full width to 3 db of $\Delta\nu$:

$$G(\nu') = \frac{\gamma}{\sqrt{\pi}\Delta\nu} \exp\left(\frac{-\gamma^2\nu'^2}{\Delta\nu^2}\right)$$

where $\gamma = 2\sqrt{\ln 2}$.

Thus $g(\tau) = \exp\left(-\pi^2 \frac{\Delta\nu^2}{\gamma^2} \tau^2\right) = \exp\left(-\pi^2 \frac{\Delta\nu^2}{\gamma^2} \frac{x_o^2u^2}{\nu_o^2}\right)$

and the transform over u gives:

$$D(x) = \frac{\gamma\nu_o}{\sqrt{\pi}\Delta\nu x_o} \exp\left(-\gamma^2 \frac{\nu_o}{\Delta\nu^2} \frac{x^2}{x_o^2}\right) \tag{9}$$

<u>Case 1</u>

Here we take a Gaussian taper as in Class A, Case 2:

$$T(u) = \frac{\sqrt{\pi}b}{\gamma} \exp\left(\frac{-\pi^2b^2u^2}{\gamma^2}\right).$$

Rather than convolve the beam with equation (9), it is somewhat simpler to multiply in the u,v plane and transform. Thus the modified visibilities are:

$$\frac{\sqrt{\pi}b}{\gamma} \exp\left[\frac{-\pi^2u^2}{\gamma^2}\left(b^2 + \frac{\Delta\nu^2}{\nu_o^2} x_o^2\right)\right]$$

which transforms to:

$$B_D(\Delta x, x_o) = \frac{b}{\sqrt{b^2 + \frac{\Delta\nu^2}{\nu_o^2} x_o^2}} \exp\left[-\frac{\Delta x^2 \gamma^2}{b^2 + \frac{\Delta\nu^2}{\nu_o^2} x_o^2}\right]$$

or, $$\boxed{B_D(\Delta x, x_o) = \frac{1}{\sqrt{1+\beta^2}} \exp\left[-\frac{\alpha^2\gamma^2}{1+\beta^2}\right]} \tag{10}$$

where all symbols have the same meaning as in the preceeding examples.

At the beam center, $\alpha = 0$, and

$$B_D(0, x_o) = \frac{1}{\sqrt{1+\beta^2}} \tag{11}$$

IV. DISPLAY OF THE DERIVED FUNCTIONS

Bandwidth loss causes two major effects: a loss in beam intensity, causing a loss in signal/noise, and a degradation of the beamwidth. In this section we consider these effects in detail for the three cases given in Section III.

A. Loss of Intensity at Beam Center

The expressions are:

A.1 No u,v taper, square bandpass

$$B_D(0, x_o) = \frac{2}{\eta\beta} \text{Si}\left(\frac{\eta\beta}{2}\right), \quad \eta = 3.79$$

A.2 Gaussian u,v taper, square bandpass

$$B_D(0, x_o) = \frac{\sqrt{\pi}}{\gamma\beta} \text{erf}\left(\frac{\gamma\beta}{2}\right), \quad \gamma = 1.665$$

B.1 Gaussian u,v taper, Gaussian bandpass

$$B_D(0, x_o) = \frac{1}{\sqrt{1+\beta^2}}$$

For all expressions,

$$\beta = \frac{\Delta\nu}{\nu_o} \cdot \frac{x_o}{b}$$

These functions are plotted in Figure 1. Note that, overall, they all are rather similar. This is especially true for the two square bandpass examples - the more realistic cases. This agreement illustrates the applicability of β as a universal parameter - the variations in beamsize due to u,v coverage and taper are absorbed in the beamwidth, b.

B. Beamshape

The expressions are:

A.1 $$B_D(\Delta x, x_o) = \frac{1}{\eta\beta} \left[Si(\eta\alpha + \frac{\eta b}{2}) - Si(\eta\alpha - \frac{\eta b}{2}) \right]$$

A.2 $$B_D(\Delta x, x_o) = \frac{\sqrt{\pi}}{2\beta\gamma} \left[erf(\gamma\alpha + \frac{\gamma b}{2}) - erf(\gamma\alpha - \frac{\gamma b}{2}) \right]$$

B.1 $$B_D(\Delta x, x_o) = \frac{1}{\sqrt{1+\beta^2}} \exp \left[- \frac{\gamma^2}{1+\beta^2} \alpha^2 \right]$$

with $\eta = 3.79$, $\gamma = 1.665$, and $\alpha = \Delta x/b$.

The functions are plotted in Figures 2, 3, and 4. Again, use of the normalized parameters, α and β, results in remarkably similar functions, considering the general range allowed in observing parameters. These three plots give the detailed beamshapes, normalized to the peak response, for the three cases.

A parameter of much interest is the broadening of the beam as measured by the half-power. This can be read off Figures 2, 3 and 4, the results of this are shown in Figure 5. The ordinate is the half-power (FWHP) in units of the undistorted beamwidth; the abscissa is the normalized parameter,

$$\beta = \frac{\Delta\nu}{\nu_o} \bullet \frac{x_o}{b} .$$

It may appear that the Gaussian taper gives better immunity from bandwidth losses - in actuality, this is due to the division by the beamwidth. For the Gaussian case to give the same beam as the no-taper case, considerably longer u,v spacings are required. These are affected by bandwidth losses more severely than the corresponding no-taper case, thus giving a worse initial broadening.

V. DISCUSSION

The most important results deriving from this work are shown in Figure 1 and Figure 5 - the loss of central intensity and broadening of the synthesized beams due to bandwidth losses. An important note is that the effects of spacing (array scale), frequency, duration of observation, and bandwidth, can all be absorbed into a single parameter, denoted by β. Use of this parameter allows approximately correct results to be derived quickly without recourse to detailed and complicated calculations.

As a final note, observe that the bandwidth effect is <u>independent of observing frequency</u>. This can most easily be realized by noting that the loss function is given by the transform of the bandpass shape, referenced to the center frequency. The fact that frequency appears in the parameter β is due solely to the simultaneous presence of the beamwidth. The product of these two is independent of frequency.

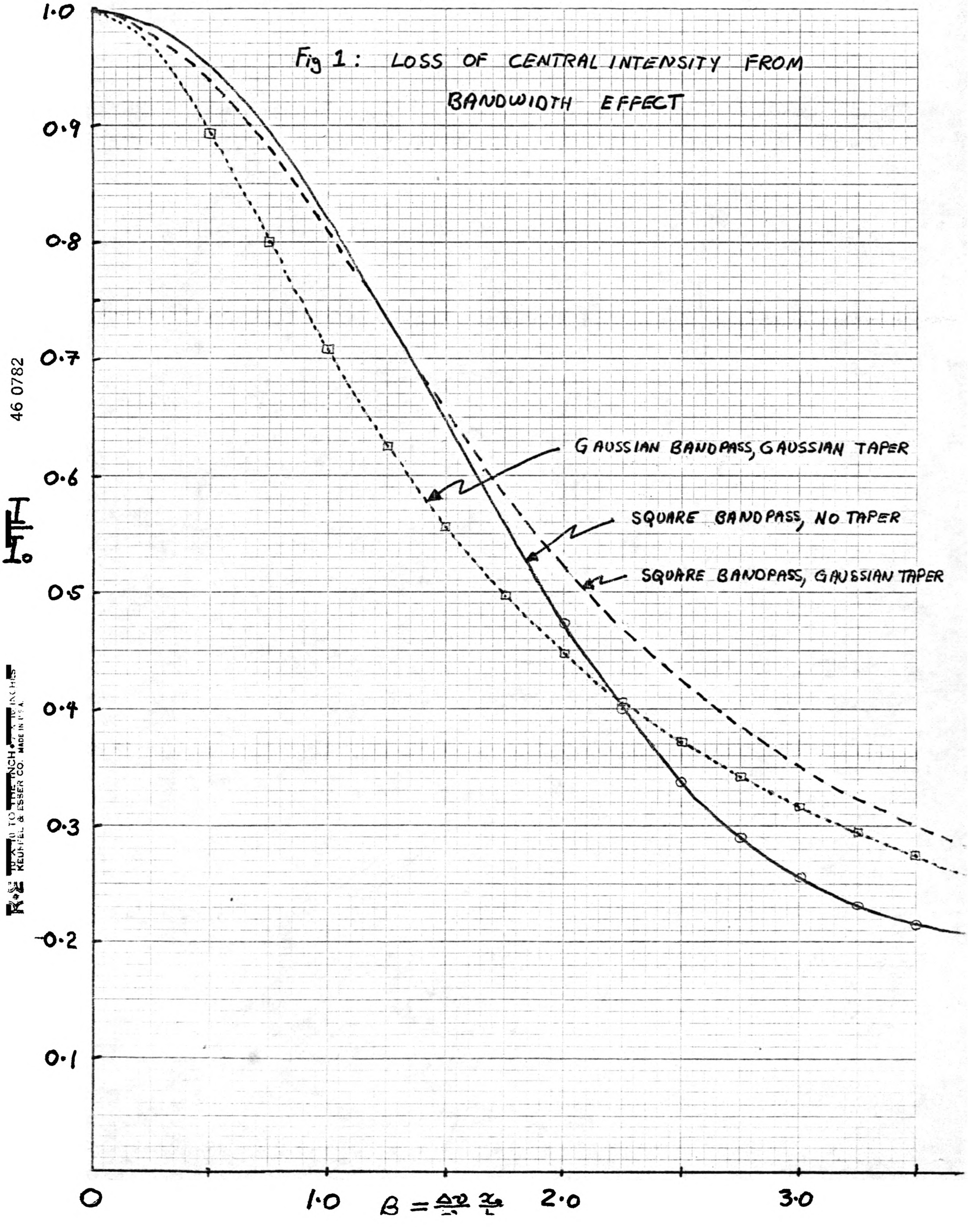
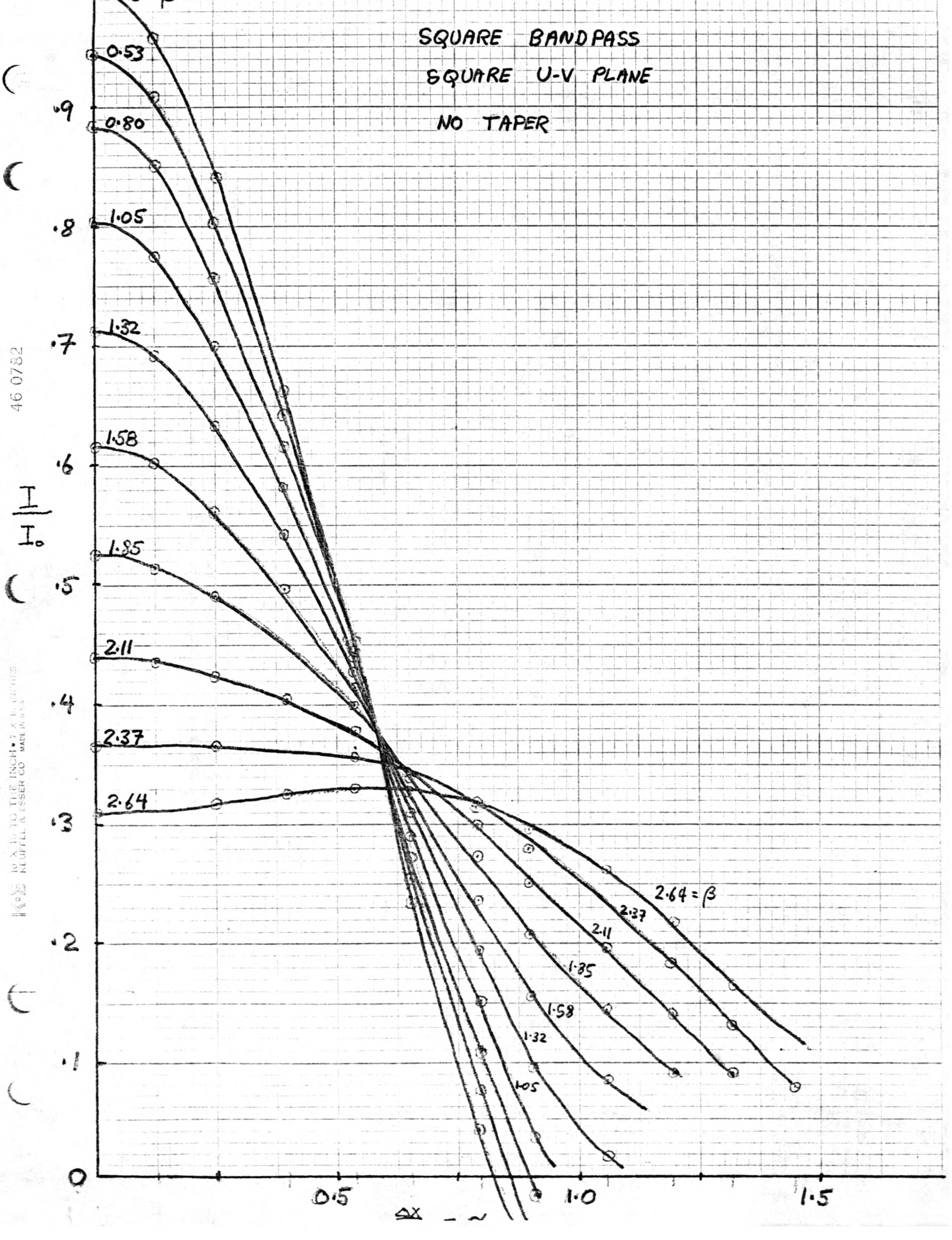
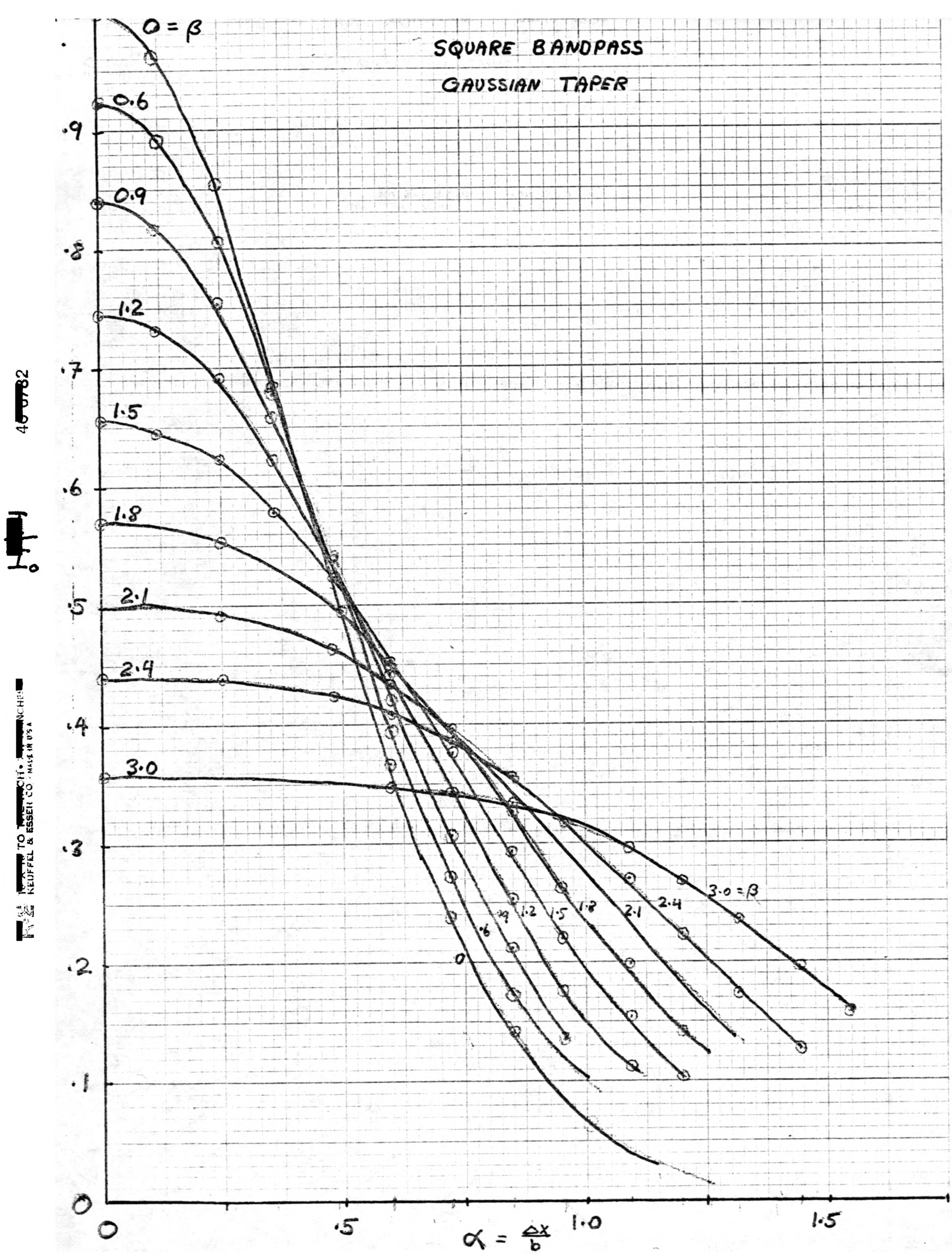


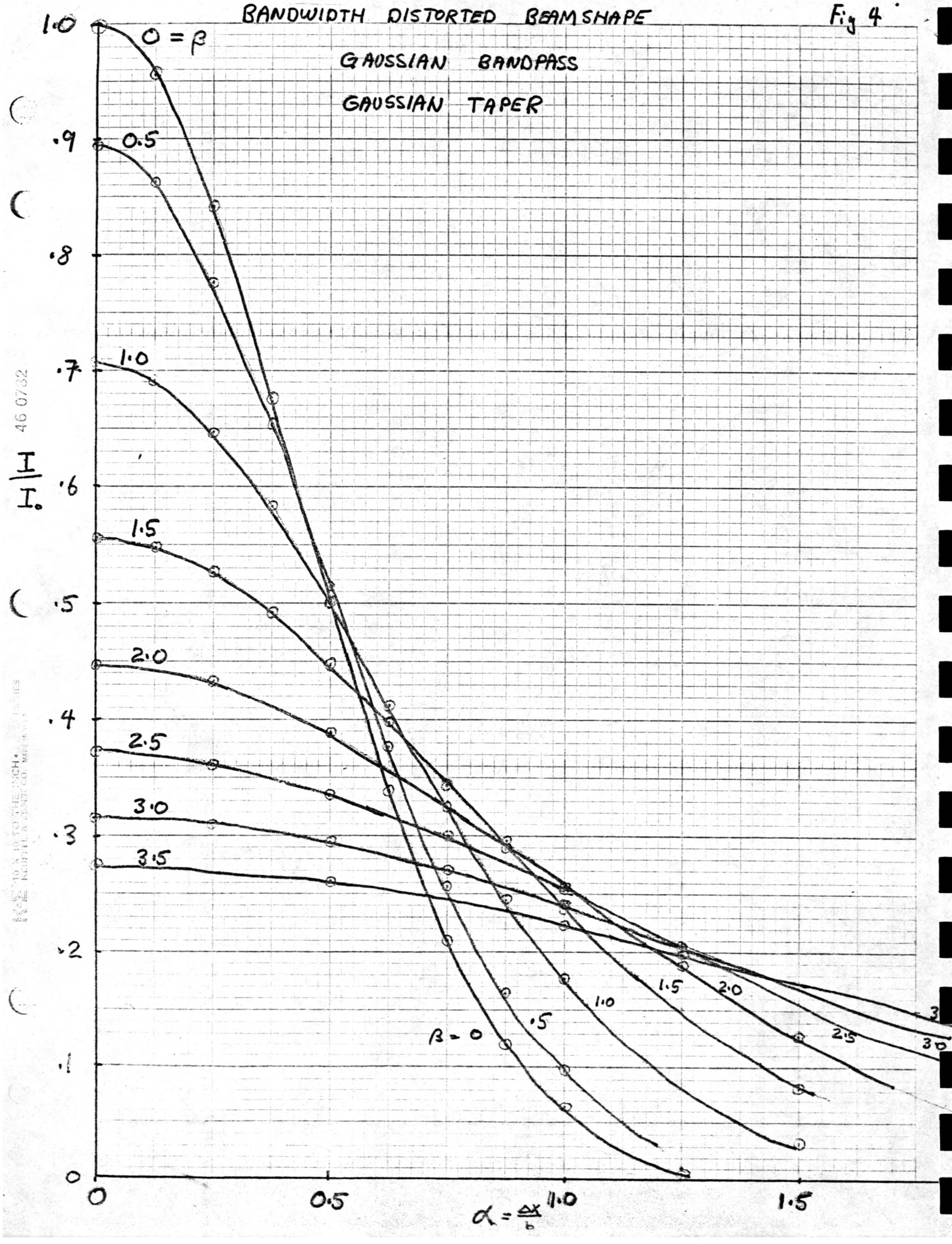
Fig 1: LOSS OF CENTRAL INTENSITY FROM BANDWIDTH EFFECT

BANDWIDTH DISTORTED BEAMSHAPE

Fig 2.




SQUARE BANDPASS
GAUSSIAN TAPER
0 = β
0.6
0.9
1.2
1.5
1.8
2.1
2.4
3.0
3.0 = β
α = Δx/b
0
·5
1·0
1·5
·1
·2
·3
·4
·5
·6
·7
·8
·9


BANDWIDTH DISTORTED BEAMSHAPE
Fig 4
GAUSSIAN BANDPASS
GAUSSIAN TAPER
0 = β
0.5
1.0
1.5
2.0
2.5
3.0
3.5
β = 0
.5
1.0
1.5
2.0
2.5
3.0
1.0
.9
.8
.7
.6
.5
.4
.3
.2
.1
0
$\frac{I}{I_o}$
0
0.5
1.0
1.5
$\alpha = \frac{\Delta x}{b}$

BEAM BROADENING BY BANDWIDTH EFFECT

Fig 5