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On the Effects of Convolution in the u-v Plane

by

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Aperture synthesis has been discussed extensively in various VLA memoranda and publications. However, there still appears to be confusion concerning the use of a convolving function in the u-v plane. Employing an artificial data producing program and the NRAO line interferometer data reduction package, I have prepared demonstrations of the effects of the convolution on source, sidelobe, and noise amplitudes.

A. The Convolution

In the classical FFT approach to aperture synthesis, the visibility function which is transformed is given by

$$V'' = III \cdot (C * (V \cdot S)) \qquad (1)$$

where

$*$ is a convolution

$S = \sum_j W_j \, {}^2\delta(u-u_j,\ v-v_j)$ = data sampling function

V = true source visibility

C = convolving function (to "smooth data to a grid")

$III = \sum_m \sum_n {}^2\delta(u-m\Delta u,\ v-n\Delta v)$

(see "The VLA Spectral Line System: A Progress Report"). The convolving function must be used to allow the data to be resampled (by III) on a rectangular grid. The resulting map is given by

$$T'' = \overline{III} * (\overline{C} \cdot (T * B)) \qquad (2)$$

where T", $\overline{III}$, $\overline{C}$, T, and B are the Fourier transforms of V", III, C, V, and S, respectively. We usually ignore the effects of aliasing (the $\overline{III}$ *) to obtain

$$T' = \frac{T''}{\overline{C}} = T * B \tag{3}$$

which is the immediate result of the direct Fourier transform.

To illustrate the operation of equation (3) (i.e., to illustrate the fact that both source and sidelobe (*B) structures must be divided by $\overline{C}(x,y)$), I have prepared Figure 1. Figure 1a illustrates the synthesized beam pattern (B) for baselines of 100 and 200 meters at Green Bank for a source at the north pole. This figure was produced by a direct Fourier transform and, hence, is not affected by any convolution function. A map of a small diameter source somewhat to the left of the phase center, synthesized with these baselines, is shown in Figure 1b. The map was produced by an FFT with a pill-box convolving function of width 3.5 cells and no division by $\overline{C}$ was performed. Note the strong effect of $\overline{C}$ on the circular sidelobe pattern. Figure 1c shows the map of Figure 1b after division by $\overline{C}$. The circular pattern is restored and the source amplitude increased (by 8%, here).

B. Noise

It is commonly assumed that the noise on maps (T") produced by the FFT is essentially independent of position and that the division by $\overline{C}$ increases the noise at the edges of the map (where $\overline{C}$ is not ~1.0). I will use both algebra and a practical demonstration to prove that these assumptions are not correct.

Let us first consider the case of smoothed data which are not resampled on a rectangular grid. Writing everything out, we are determining the rms of T(x,y) which is given by

$$T(x,y) = \kappa \sum_{j=-N}^{N} W_j \iint_{-\infty}^{\infty} du\,dv\; e^{-2\pi i(ux+vy)} \iint_{-\infty}^{\infty} du'dv' \; V_j \; {}^2\delta(u'-u_j,\; v'-v_j)\; C\,(u'-u,\; v'-v) \tag{4}$$

where C is the convolving function and the V_j are the data samples. Using

$$\sigma^2 = \left\langle \left| \sum_j \frac{\partial T}{\partial V_j} \Delta V_j \right|^2 \right\rangle \quad ,$$

we get

$$\sigma^2(x,y) = \kappa^2 \sum_{j=-N}^{N} \left\langle |\Delta V_j|^2 \right\rangle \quad W_j^{\ 2}$$

$$\left| \iint_{-\infty}^{\infty} dudv \; e^{-2\pi i(ux+vy)} \iint_{-\infty}^{\infty} du'dv' \; C(u'-u,v'-v) \; {}^2\delta(u'-u_j, \; v'-v_j) \right|^2$$

where we have ignored, with the usual justifications, terms in $<\Delta V_j \Delta V_k^*>$ and $<\Delta V_j \Delta V_j>$. We can do the primed integral now to get

$$\sigma^2 = \kappa^2 \sum W_j^{\ 2} \left\langle |\Delta V_j|^2 \right\rangle \left| \iint dudv \; e^{-2\pi i(ux+vy)} \; C(u_j-u, \; v_j-v) \right|^2 . \quad (5)$$

Substituting $u' = u_j-u$ and $v' = v_j-v$, we can do the unprimed integral (which is $\overline{C}(x,y)$ times a phase term) to get

$$\sigma(x,y) = |\overline{C}\;(x,y)| \left[\sum_{j=-N}^{N} \kappa^2 \; W_j^{\ 2} \left\langle |\Delta V_j|^2 \right\rangle \right]^{1/2}$$

Thus, the expected noise is, spatially, a constant multiplied by the absolute value of the Fourier transform of the convolving function. In other words, the signal-to-noise ratios are not affected by the convolution.

For completness, let us also consider the true FFT situation in which the sampled visibility function is smoothed and then resampled by the III function. In this case, the interesting portion of equation (5) becomes

$$\left| \iint dudv \sum_m \sum_n {}^2\delta(u-m\Delta u, \; v-n\Delta v) e^{-2\pi i(ux+vy)} \; C(u_j-u, v_j-v) \right|^2$$

$$= \left| \sum_m \sum_n \iint du'dv' \; C(u',v') \; e^{2\pi i(u'x+v'y)} \; {}^2\delta(u_j-u'-m\Delta u, \; v_j-v'-n\Delta v) \right|^2$$

$$= \left| \sum_m \sum_n \; C(u_j-m\Delta u, \; v_j-n\Delta v) \; e^{-2\pi i(m\Delta ux+n\Delta vy)} \right|^2$$

This result is $\overline{C}(x,y)$ only if u_j equals some $m\Delta u$ and v_j equals some $n\Delta v$. Thus aliasing also complicates our noise analysis by introducing a dependence on the positions of the data samples.

For those readers who are distrustful of long algebraic formulae, I have also studied this subject empirically. I have used an artificial data program to observe a "source-free" region with the same u-v coverage used in Figure 1 and with 50 channels of randomly-generated noise (@ 1 Jy/ 80-second interval). The fifty channels were Fourier transformed (separately) by an FFT with a 3.5 x 3.5 cell pill-box convolving function both with and without division by $\overline{C}$. The two sets of 50 maps were then used to determine point-by-point rms maps. Figure 2a shows the rms map without division by $\overline{C}$ and Figure 2b shows the rms map with division by $\overline{C}$. The average rms of Figure 2a is 0.77°K with an rms of 0.49°K while the average rms of Figure 2b is 1.74°K with an rms of 0.20°K. The effects of $\overline{C}$ should be obvious to all.

C. Conclusion

I have examined the effects of convolution in the u-v plane on the source, sidelobe, and noise amplitudes in the map plane. It is clear that we must divide the output maps by the Fourier transform of the convolving function in order to obtain correct amplitudes and that this division does not affect the signal-to-noise ratios. I should also point out that we must divide output "cleaned" maps by the power pattern of the averaged single-dish antenna. This latter division is also needed to determine correct amplitudes, but it will cause the noise to be greater in the outer portions of the map area.

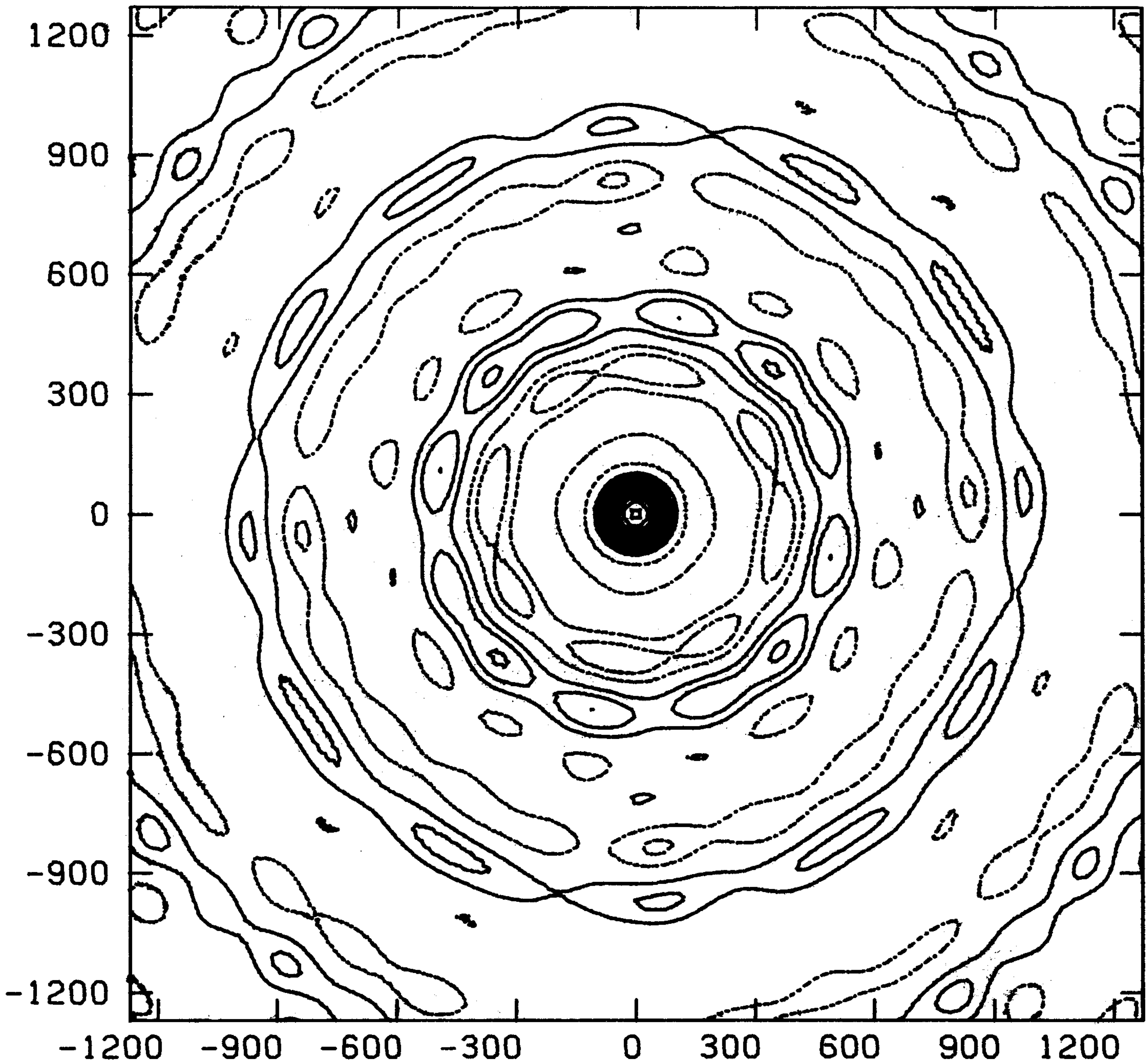

Figure 1a.
Synthesized beam pattern by direct Fourier transform. The contour interval is 10% of the peak with negative contours shown as broken lines and with the zero contour suppressed.

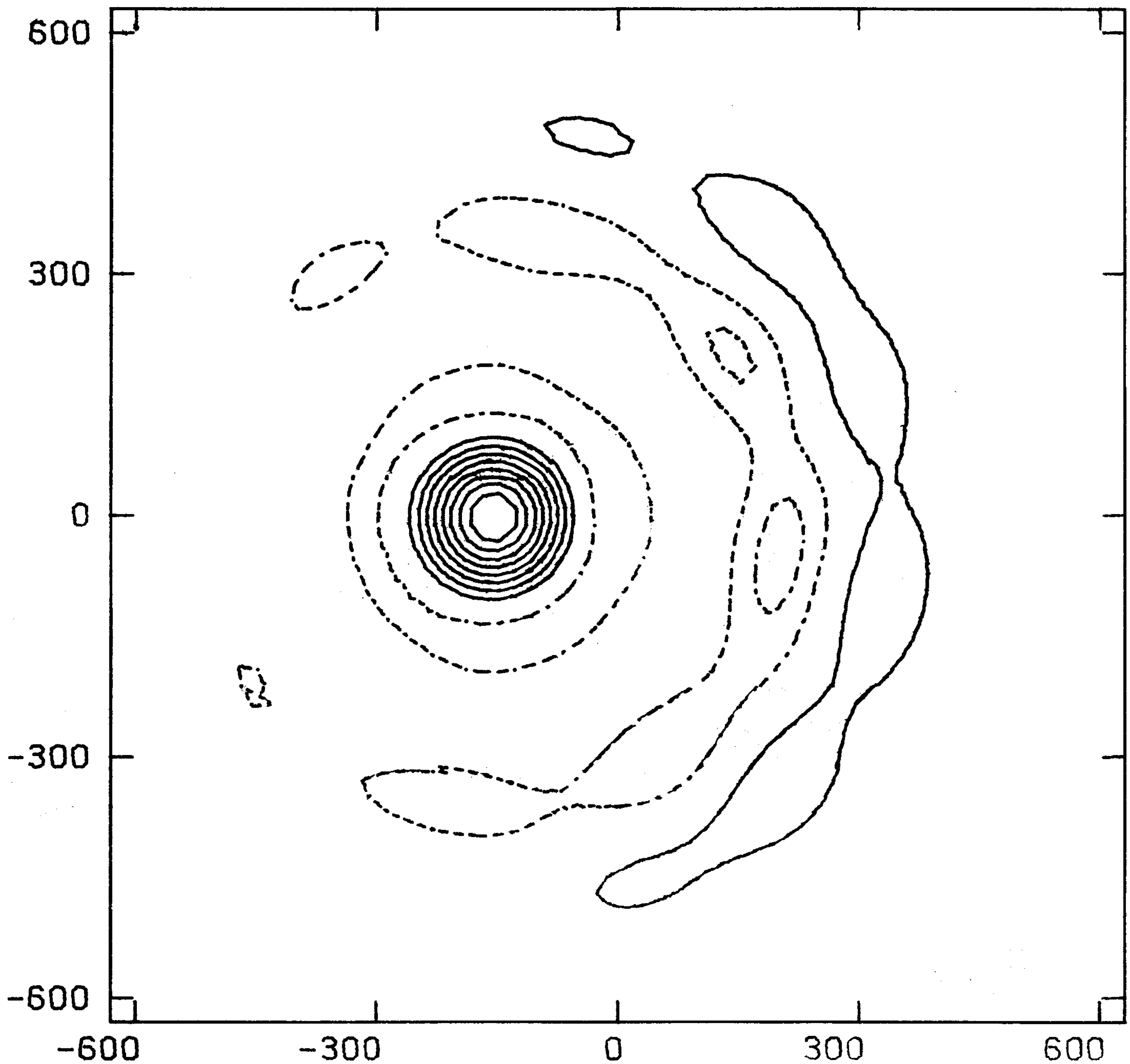

Figure 1b.
Synthesized map by FFT with no correction for convolution function. The contour interval is 100°K with negative contours shown as broken lines and with the zero contour suppressed.

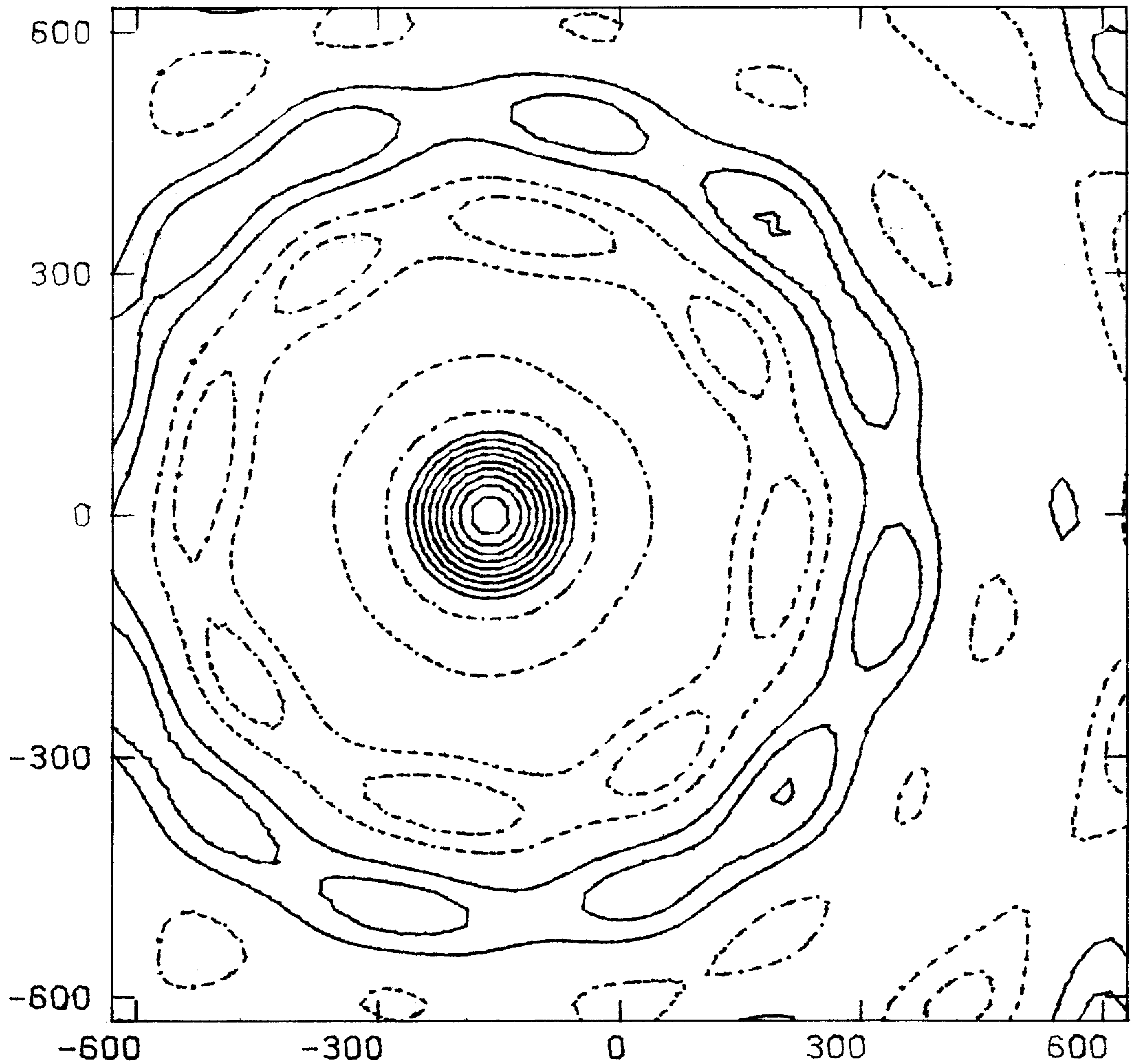

Figure 1c.
Synthesized map by FFT with correction for convolution function. The contour interval is 100°K with negative contours shown as broken lines and with the zero contour suppressed.

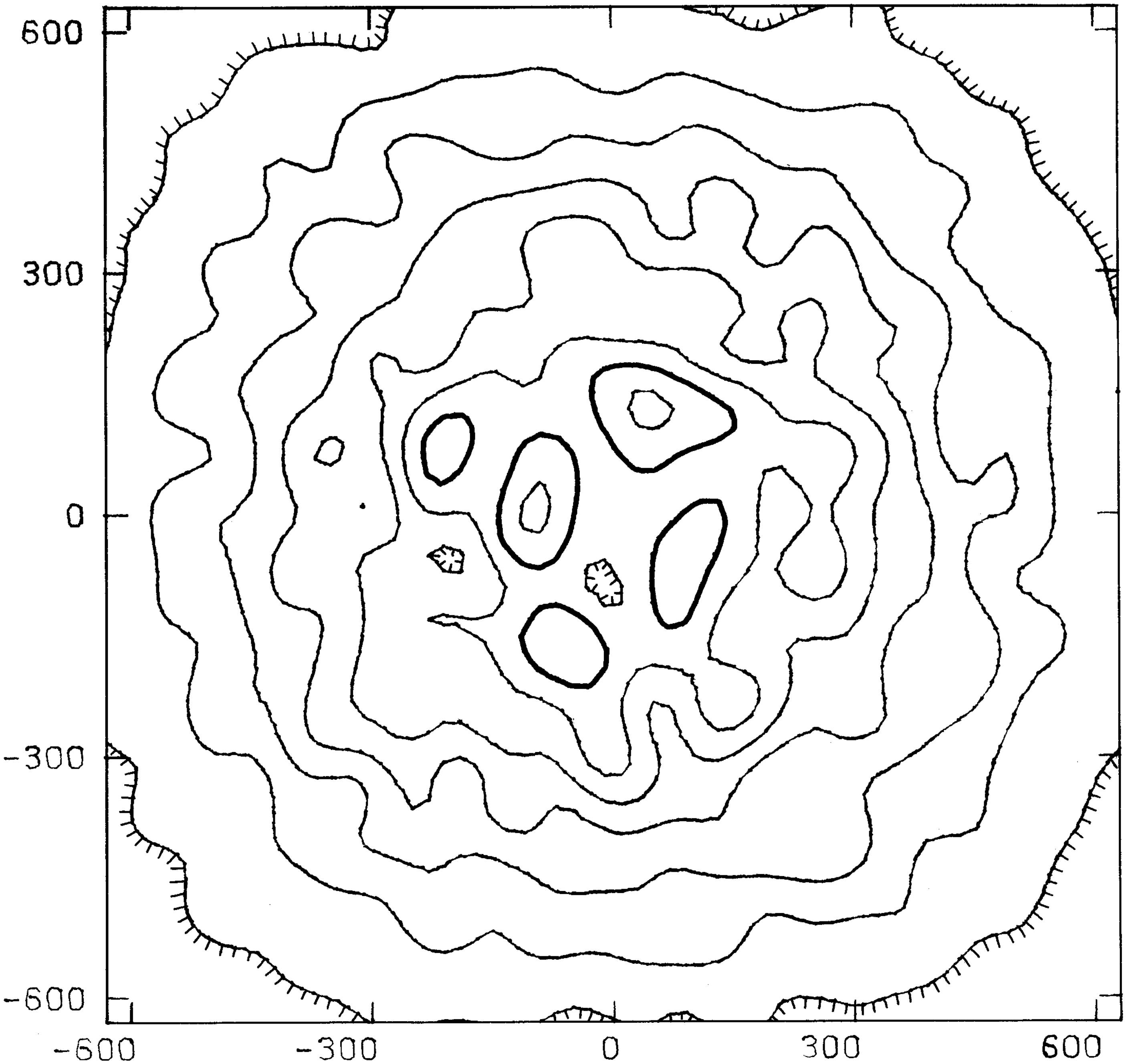

Figure 2a.

Map of the rms of 50 source-free maps produced by FFT with no correction for convolution function. The contour interval is 0.25°K and the 1.75°K contour is shown as a heavy line.

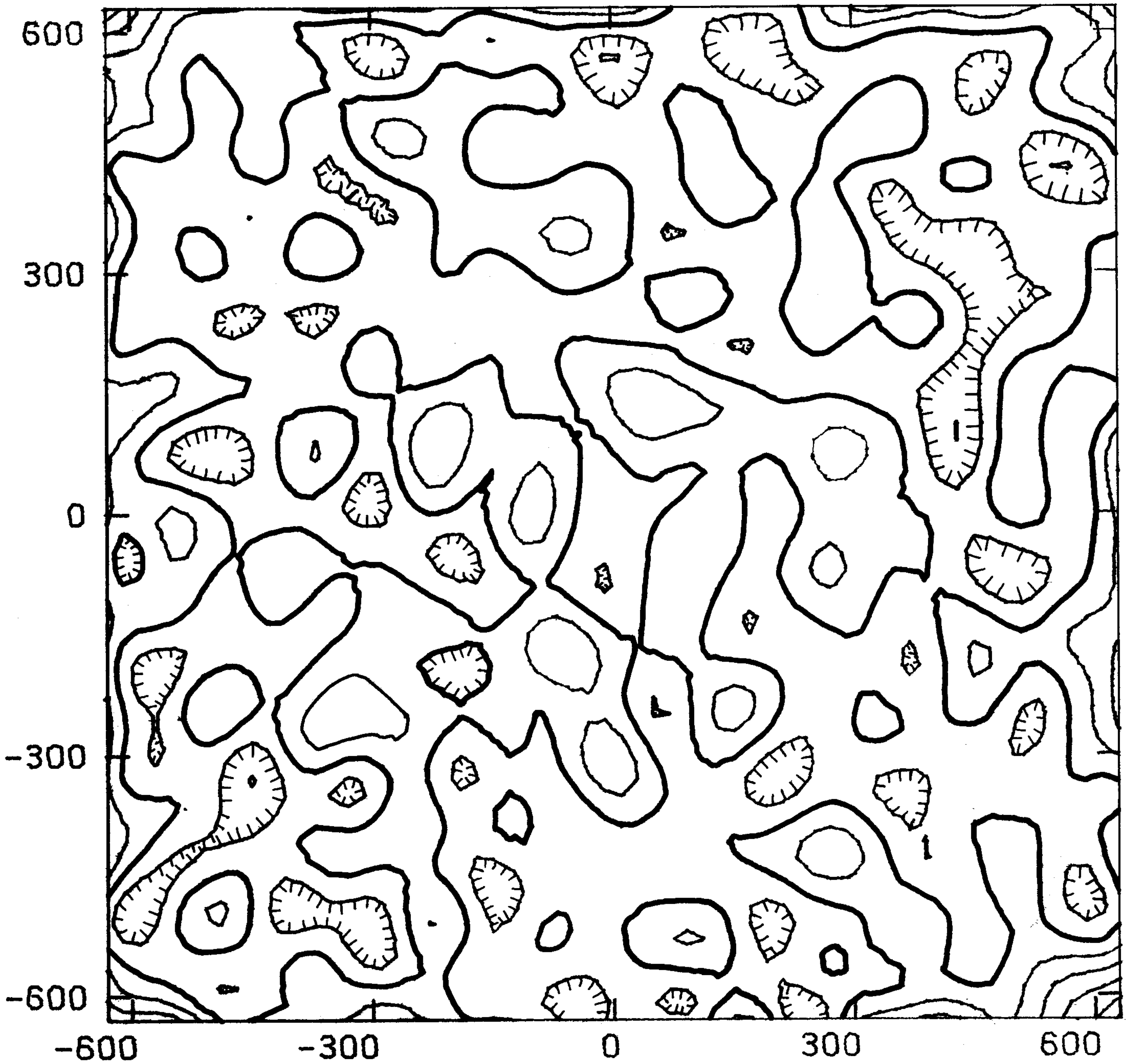

Figure 2b

Map of the rms of 50 source-free maps produced by FFT with correction for convolution function. The contour interval is 0.25°K and the 1.75°K contour is shown as a heavy line.