Effect of Threshold Errors on the Response of FX Correlator. (Low Input Correlation)

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1 Introduction

The two and four levels quantization schemes have been addopted in the design of the VLBA recording system. The conversion functions between input correlation of the analog signal and measured correlation of digitized signal are well known for all three possible combinations of the digitizers: TWO-TWO, TWO-FOUR, and FOUR-FOUR [1], [2], [3]. The parameters of the four levels digitizer: threshold v and ratio of upper levels n are chosen close to the optimal values v= 0.9816, n=3.336 [2], [3]. In the real life it is difficult to have the ideal simmetrical digitizers with the exactly optimal value of its parameters. In fact the real digitizer is not simmetrical due to DC offset and the thresholds differ from optimal ones. The problem was formulated by D. Bagri [4]. In this memo we'll try to estimate the influence of the errors of the digitizers levels on the output of FX correlator for the three possible combinations of the digitizers: FOUR-FOUR, TWO-FOUR and TWO-TWO.

2 The conversion function analysis

The operation of four level quantizer of the signal is described by a step function $q(\varepsilon)$:

$$q(\varepsilon) = \begin{cases} +n & \text{if} \quad \varepsilon \geq a \cdot \sigma \\ +1 & \text{if} \quad 0 \leq \varepsilon < a \cdot \sigma \\ -1 & \text{if} \quad b \cdot \sigma \leq \varepsilon < 0 \\ -n & \text{if} \quad \varepsilon < b \cdot \sigma \end{cases} \tag{1}$$

where σ is the variance of the input analog random process $\varepsilon(t)$,
a, b are the upper and low thresholds of the digitizer measured at number of σ,
n is the ratio of the levels.

In an ideal case $a = -b = v = 0.9816$, n =3.336 for both site's recording system. In fact the thresholds a_1, a_2, b_1, and b_2 are different:

$$\begin{array}{llll} a_1 = & v+ & \alpha_1 = & v + \epsilon_1 + \delta_1 \\ a_2 = & v+ & \alpha_2 = & v + \epsilon_2 + \delta_2 \\ b_1 = - & v+ & \beta_1 = - & v - \epsilon_1 + \delta_1 \\ b_2 = - & v+ & \beta_2 = - & v - \epsilon_2 + \delta_2 \end{array} \tag{2}$$

where δ_1, δ_2 are DC offsets
ϵ_1, ϵ_2 are input power level (sampler gain) changes.

Following Schwab [2], we can obtain the next formula for the expected value $r(\rho)$ of the correlator output for the case of low input correlation:

$$r(\rho) = r(0) + \{(n-1)^2[g(a_1,a_2) + g(b_1,b_2) + g(a_1,-b_2)] + 2(n-1)[g(a_1,0) + g(b_1,0) + g(a_2,0) + g(b_2,0)] + \tfrac{2}{\pi}\}\rho \tag{3}$$

where $g(x,y) = \frac{1}{2\pi}e^{-\frac{x^2+y^2}{2}}$.
Using (2) and (3) and providing a simple analysis of TWO-TWO case, we can obtain the next formulas for the deviation of output correlation for FOUR-FOUR, TWO-FOOR, and TWO-TWO digitizers combinations (leaving only the first terms of the errors):

$$\Delta r_{44} = r_{44}(0) - \frac{2}{\pi}(n-1)ve^{-\frac{v^2}{2}}\left[(n-1)e^{-\frac{v^2}{2}} + 1\right](\epsilon_1 + \epsilon_2)\rho \tag{4}$$

$$\Delta r_{24} = r_{24}(0) - \frac{2}{\pi}n(n-1)ve^{-\frac{v^2}{2}}\epsilon\rho \tag{5}$$

$$\Delta r_{22} = r_{22}(0) - \frac{2}{\pi}\frac{1}{2}(c_1^2 + c_2^2)\rho \tag{6}$$

where c_1, c_2 are errors of determination of zero line at a two levels' digitizers.
The expression for $r_{44}(0)$, $r_{24}(0)$ and r_{22} can be written as (Appendix):

$$r_{44}(0) = (n-1)^2\frac{2}{\pi}e^{-v^2}\delta_1\delta_2 \tag{7}$$

$$r_{24}(0) = (n+1)\frac{2}{\pi}e^{-\frac{v^2}{2}}c\,\delta \tag{8}$$

$$r_{22}(0) = \frac{2}{\pi}c_1\,c_2 \tag{9}$$

Having substituted the optimal values v=0.98 and n=3.336 into equations (7), (8), (9) we obtain the final expression for DC offset of measured correlation:

$$r_{44}(0) = 1.32\,\delta_1\delta_2, \quad r_{24}(0) = 1.70\,c\,\delta, \quad r_{22}(0) = 0.64\,c_1c_2 \tag{10}$$

3 Discussion

So the effect of the threshold errors on the correlation measurements consist in DC offset and changing of the conversion function slope. The effect of DC is weakened by averaging at the stage of fringe stopping. Indeed, r.m.s. of measured correlation and DC offset at output of fringe stopping are equal:

$$\begin{array}{lll} r.m.s.(r) \approx \frac{r_{max}}{\sqrt{\Delta f\tau}}, & DC = \frac{1.32\delta_1\delta_2}{f_r\tau} & FOUR-FOUR \\ r.m.s.(r) \approx \frac{r_{max}}{\sqrt{\Delta f\tau}}, & DC = \frac{1.70c\delta}{f_r\tau} & TWO-FOUR \\ r.m.s.(r) \approx \frac{r_{max}}{\sqrt{\Delta f\tau}}, & DC = \frac{0.64c_1c_2}{f_r\tau} & TWO-TWO \end{array} \tag{11}$$

where Δf is the bandwidth of correlating signals,
f_r is the fringe rate,
τ is the time of coherent averaging,
r_{max} is the maximum value of correlation of the digitized signals;
This value is equal 4.3048, 5.8784 and 1.0 for FOUR-FOUR, TWO-FOUR and TWO-TWO cases respectively [1], [2], [3].

We can neglect the DC offset if $DC \ll r.m.s.(r)$ i.e. the next unequalities are satisfyed:

$$\begin{array}{ll} \delta_1\delta_2 \ll 3.26\sqrt{\frac{f_r}{\Delta f}}\sqrt{f_r\,\tau} & FOUR-FOUR \\ c\,\delta \ll 3.46\sqrt{\frac{f_r}{\Delta f}}\sqrt{f_r\,\tau} & TWO-FOUR \\ c_1c_2 \ll 1.57\sqrt{\frac{f_r}{\Delta f}}\sqrt{f_r\,\tau} & TWO-TWO \end{array} \tag{12}$$

Let's substitute at the previous unequalities the bandwidth as large as $\Delta f \approx 100\,MHz = 10^8\,Hz$, the fringe rate as small as $10\,Hz$, and the time of averaging 100sec. Then we obtain the next reqirement to the DC offset of the digitizers: .

$$\begin{array}{ll} \sqrt{\delta_1\delta_2} \ll 0.181 = 18.1\% & FOUR-FOUR \\ \sqrt{c\delta} \ll 0.186 = 18.6\% & TWO-FOUR \\ \sqrt{c_1c_2} \ll 0.125 = 12.5\% & TWO-TWO \end{array} \tag{13}$$

These conditions can be satisfied rather simple.
So generally the effect of DC offset is negligible.
Taking it into account and using (4), (5), (6), we can calculate the relative error of the conversion function's estimation (error of its slope):

$$\frac{\Delta r_{44}}{r_{44}} = \frac{v}{1+(n-1)^{-1}e^{\frac{v^2}{2}}}(\epsilon_1+\epsilon_2) \tag{14}$$

$$\frac{\Delta r_{24}}{r_{24}} = \frac{v}{1+(n-1)^{-1}e^{\frac{v^2}{2}}}\epsilon \tag{15}$$

$$\frac{\Delta r_{22}}{r_{22}} = \frac{1}{2}(c_1^2+c_2^2) \tag{16}$$

The necessary expressions for r_{44}, r_{24} and r_{22} have been taken from [3]. Having substituted the optimal values v=0.98 and n=3.336 into equations (14, 15), we can obtain the expresion connecting the error of the gains ϵ and the relative error of the conversion function's estimation:

$$\frac{\Delta r_{44}}{r_{44}} = 0.6(\epsilon_1+\epsilon_2) \tag{17}$$

$$\frac{\Delta r_{24}}{r_{24}} = 0.6\epsilon \tag{18}$$

So, we have to know the gain with accuracy better than 0.075% (FOUR-FOUR) and 0.15 % (TWO-FOUR) to garanty the accuracy of correlation estimation better than 0.1%. At the TWO-TWO case error of the slope of the conversion function does not include the first power of the zero line errors c_1, c_2 (see equation (16). So this digitizer's combination provides a minimum problem comparatevely with TWO-FOUR and FOUR-FOUR cases. In the case of low correlation the errors of threshold's estimation leads to changing all correlation coefficients by the same factor.
So the cross correlation spectum has been changed only by the scale but not by the shape.

APPENDIX
DC TERM AT THE CONVERSION FUNCTION.

FOUR-FOUR

The expression for the correlation of the four levels digitized signals $r_{44}(\rho)$ is represented by the next equation [2]:

$$\begin{aligned} r_{44}(\rho) = \quad & (n-1)^2[L(a_1,a_2,\rho)+L(a_1,b_2,\rho)+L(b_1,a_2,\rho)+L(b_1,b_2,\rho)+1] \\ & +2(n-1)[L(a_1,0,\rho)+L(b_1,0,\rho)+L(a_2,0,\rho)+L(b_2,0,\rho)] \\ & -n(n-1)[Q(a_1)+Q(b_1)+Q(a_2)+Q(b_2)]+\tfrac{2}{\pi}\arcsin\rho \end{aligned} \tag{19}$$

where $L(h,k,\rho)=\int_h^\infty\int_k^\infty \frac{1}{2\pi\sqrt{1-\rho^2}}e^{-\frac{1}{2}\left(\frac{x^2-2\rho xy+y^2}{1-\rho^2}\right)}dx\,dy$

$Q(x)=\frac{1}{\sqrt{2\pi}}\int_x^\infty e^{\frac{-t^2}{2}}dt$

If $\rho=0$ then the equation (19) can be simplifyed:

$$\begin{aligned} r_{44}(0) = \quad & (n-1)^2[Q(a_1)Q(a_2)+Q(a_1)Q(b_2)+Q(b_1)Q(a_2)+Q(b_1)Q(b_2)+1] \\ & +(n-1)[Q(a_1)+Q(b_1)+Q(a_2)+Q(b_2)] \\ & -n(n-1)[Q(a_1)+Q(b_1)+Q(a_2)+Q(b_2)] \end{aligned} \tag{20}$$

Let's mark $S_1=Q(a_1)+Q(b_1)$, $\quad S_2=Q(a_2)+Q(b_2)$. Then the equation (20) can be simplifyed:

$$r(\rho)=(n-1)^2[(1-S_1)(1-S_2)] \tag{21}$$

Having used the definition of the function $Q(x)$ we can obtain the next formula for S1 and S2:

$$S_1=1+\int_{a_1}^{b_1}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx, \qquad S_2=1+\int_{a_2}^{b_2}\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx \tag{22}$$

The deviation of real thresholds a_1,a_2,b_1,b_2 from ideal values v or $-v$ is very small:

$$\begin{array}{llll} a_1= & v+ & \alpha_1= & v+\epsilon_1+\delta_1 \\ a_2= & v+ & \alpha_2= & v+\epsilon_2+\delta_2 \\ b_1=- & v+ & \beta_1=- & v-\epsilon_1+\delta_1 \\ b_2=- & v+ & \beta_2=- & v-\epsilon_2+\delta_2 \end{array} \tag{23}$$

So we can take the integrals at (22) considering $e^{-\frac{x^2}{2}}=e^{-\frac{v^2}{2}}$ for the whole interval of integrating:

$$S_1=1+\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}2\delta_1, \qquad S_2=1+\frac{1}{\sqrt{2\pi}}e^{-\frac{v^2}{2}}2\delta_1 \tag{24}$$

Having substituted (24) in (21) we obtain the final formula for DC offset of the measured correlation:

$$r_{44}(0)=(n-1)^2\frac{2}{\pi}e^{-v^2}\delta_1\delta_2 \tag{25}$$

TWO-FOUR

Let's consider a, b the thresholds of a four levels digitizer and c the threshold of a two levels digitizer:

$$\begin{array}{llll} a= & v+ & \alpha= & v+\epsilon+\delta \\ b=- & v+ & \beta=- & v-\epsilon+\delta \end{array} \tag{26}$$

The expression of the correlation of so digitized signals can be represented by the next equations in condition that $\rho = 0$:

$$\begin{aligned}\rho_{24}(0) = \quad & n(\int_c^\infty g(x)\,dx \int_a^\infty g(y)\,dy - \int_c^\infty g(x)\,dx \int_{-\infty}^b g(y)\,dy \\ & + \int_{-\infty}^c g(x)\,dx \int_{-\infty}^b g(y)\,dy - \int_{-\infty}^c g(x)\,dx \int_a^\infty g(y)\,dy) \\ & +(\int_c^\infty g(x)\,dx \int_0^a g(y)\,dy - \int_c^\infty g(x)\,dx \int_b^0 g(y)\,dy \\ & + \int_{-\infty}^c g(x)\,dx \int_b^0 g(y)\,dy - \int_{-\infty}^c g(x)\,dx \int_0^a g(y)\,dy) \\ & = n\left(\int_c^\infty g(x)\,dx \int_{-b}^a dy - \int_{-\infty}^c g(x)\,dx \int_{-b}^a g(y)\,dy\right) \\ & + \left(\int_c^\infty g(x)\,dx \int_0^a g(y)\,dy - \int_{-\infty}^c g(x)\,dx \int_{-b}^a g(y)\,dy\right) \\ & = (n+1)\left(\int_{-c}^c g(x)\,dx \int_{-b}^a g(y)\,dy\right)\end{aligned} \tag{27}$$

where $g(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$

The values of α, β, c are very small comperatively with unit. So we can take the integrals at (27) considering $e^{-\frac{x^2}{2}} = e^{-\frac{y^2}{2}}$ and $e^{-\frac{x^2}{2}} = 1$ for the whole interval of integrating. Finally we obtain the next expression for $\rho_{24}(0)$:

$$r_{24}(0) = (n+1)\frac{2}{\pi} e^{-\frac{y^2}{2}} c\,\delta \tag{28}$$

TWO-TWO

Let's consider c_1, and c_2 the thresholds of two levels digitizers and

The expression of the correlation of so digitized signals can be represented by the next equations in condition that $\rho = 0$:

$$\begin{aligned}\rho_{22}(0) = \quad & (\int_{c_1}^\infty g(x)\,dx \int_{c_2}^\infty g(y)\,dy - \int_{c_1}^\infty g(x)\,dx \int_{-\infty}^{c_2} g(y)\,dy \\ & + \int_{-\infty}^{c_1} g(x)\,dx \int_{-\infty}^{c_2} g(y)\,dy - \int_{-\infty}^{c_1} g(x)\,dx \int_{c_2}^\infty g(y)\,dy) \\ & = \left(\int_{-c_1}^{c_1} g(x)\,dx \int_{-c_2}^{c_2} g(y)\,dy\right)\end{aligned} \tag{29}$$

where $g(t) = \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}$

The values of c_1, c_2 are very small comperatively with unit. So we can take the integrals at (29) considering $e^{-\frac{x^2}{2}} = e^{-\frac{y^2}{2}}$ =1 for the whole interval of integrating. Finally we obtain the next expression for $\rho_{22}(0)$:

$$r_{22}(0) = \frac{2}{\pi} c_1\, c_2 \tag{30}$$

References

[1] Van Vleck, J.H. Radio Res. Lab., Harvard University Cambridge, Mass., Rept N51, July 1943.

[2] Scwab, F.R. VLBA correlator memo 75, 1986

[3] Kogan, L.R VLBA scientific memo 5, 1993

[4] Bagri, D.S.. VLBA correlator memo 105, 1993