Multi-Level Fringe Rotation

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Station-based fringe rotation using a multi-level approximation to the trigonometric functions is an attractive secondary advantage of the FX or spectral correlator algorithm. The problems precluding such an operation in a conventional lag correlator are mentioned briefly in VLBA Correlator Memo 60, and some will become evident later in this document.

Intuitively it is obvious enough that, with sufficiently many levels and sufficiently precise specification of the transition points, an arbitrarily good approximation can be achieved. This memorandum is aimed at quantifying how good is good enough, and considering some specific approximation functions for practical use.

After a brief initial derivation which serves primarily to introduce the notation used, I describe the optimization criterion chosen and the procedure adopted, consider some important secondary effects, and discuss the characteristics of the optimized approximations.

DERIVATION

Let $G(t,\alpha)$ be a periodic, variable-width boxcar function, of period T:

$$G(t,\alpha) = \begin{cases} 1, & (k-\frac{\alpha}{4})T \le t \le (k+\frac{\alpha}{4})T \quad (\forall k); \\ 0, & \text{otherwise.} \end{cases} \tag{1}$$

For purposes of this discussion, the width parameter α is restricted to the range $[0,1]$. The complex Fourier series for $G(t,\alpha)$ is

$$G(t,\alpha) = \sum_{\forall n} g_n(\alpha) e^{i2\pi nt/T}; \tag{2}$$

$$g_n(\alpha) = \frac{1}{T}\int_0^T G(t,\alpha)e^{-i2\pi nt/T}dt = \frac{\sin\frac{\pi}{2}n\alpha}{\pi n}. \tag{3}$$

Construct a variable-duty-cycle, complex Hermitian square wave, from suitably time-translated combinations of $G(t,\alpha)$:

$$H(t,\alpha) = G(t,\alpha) + iG(t-\tfrac{1}{4}T,\alpha) - G(t-\tfrac{1}{2}T,\alpha) - iG(t-\tfrac{3}{4}T,\alpha), \tag{4}$$

with Fourier coefficients

$$h_n(\alpha) = \left[1 + ie^{-i\frac{1}{2}\pi n} - e^{-i\pi n} - ie^{-i\frac{3}{2}\pi n}\right] g_n(\alpha) = \begin{cases} \frac{4}{\pi n}\sin\frac{\pi}{2}n\alpha, & n = 1+4k \ (\forall k); \\ 0, & \text{otherwise.} \end{cases} \tag{5}$$

Then specify the multi-level rotator function — and its Fourier coefficients — as a sum of L such square waves:

$$R(t) = \sum_{\ell=1}^{L} H(t, \alpha_\ell); \qquad r_n = \sum_{\ell=1}^{L} h_n(\alpha_\ell). \tag{6}$$

The individual widths α_ℓ now form the transition points between adjacent levels, and are indexed in *decreasing* sequence,

$$\alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_L; \quad \alpha_{L+1} = 0. \tag{7}$$

This simplifies slightly the expression for the power in the waveform $R(t)$,

$$P^2 = \frac{1}{T} \int_0^T |R(t)|^2 \, dt = \frac{8}{T} \sum_{\ell=1}^{L} \ell^2 \int_{\alpha_{\ell+1}\frac{T}{4}}^{\alpha_\ell \frac{T}{4}} dt = 2 \sum_{\ell=1}^{L} (2\ell - 1)\alpha_\ell. \tag{8}$$

And normalizing the r_n by P, finally

$$\rho_n = \frac{r_n}{P} = \frac{\frac{4}{\pi n} \sum_{\ell=1}^{L} \sin \frac{\pi}{2} n \alpha_\ell}{\sqrt{2 \sum_{\ell=1}^{L} (2\ell - 1)\alpha_\ell}}, \tag{9}$$

where now the restriction of non-zero ρ_n to $n = 1 + 4k$ $(\forall k)$ is left implicit. Note that

$$L = 1, \quad \alpha_1 = \tfrac{3}{4} \qquad \Longrightarrow \qquad \rho_1 = \tfrac{4}{\pi}\sqrt{\tfrac{2}{3}} \sin \tfrac{3\pi}{8} = 0.960, \tag{10}$$

recovering the well-known result for the conventional 3-level rotator.

Optimization & Quantization

Only the fundamental note of $R(t)$ — weighted by r_1 — accomplishes the desired frequency shift. Other harmonics present in $R(t)$ shift the input spectrum to frequencies where (at best!) it does not correlate and is lost. (Worse cases are discussed later.) Thus I define the optimal set of α_ℓ as that which maximizes the *fraction of power* in the fundamental, *i.e.*, $|\rho_1|^2$. I used the IMSL routine ZXMIN, an iterative quasi-Newton multivariate extremization procedure, to solve for these optimal α_ℓ. The easily evaluated least-square fit, $\cos \frac{\pi}{2}\alpha_\ell = (\ell - \frac{1}{2})/L$, served as a convenient starting guess.

The values α_ℓ resulting from this optimization are, unfortunately, too good to be true. In a practical fringe rotator not only the number of levels, but also the precision with which the rotator phase is specified, are limited. Thus the optimal α_ℓ must be quantized onto one of M equi-spaced points spanning $[0, 1]$. At the time this calculation was programmed, we had no canned procedure available which incorporated such constraints directly (although evidently the newly-acquired NAG library includes some such). Accordingly, I adopted the following *ad hoc* approach.

After optimization, each individual parameter α_ℓ was shifted temporarily to the nearest quantization point, and the value of $|\rho_1|^2$ re-evaluated for each case. The parameter whose quantization caused the least reduction in $|\rho_1|^2$ was then fixed at the quantized value, and removed from the variational problem. The entire optimization/quantization sequence was then repeated, with one fewer free parameter, until all the α_ℓ were quantized. These then determined the final $|\rho_1|^2$ reported for the configuration defined by L and M.

For most efficient utilization of the bits available to specify these parameters, $N_{\rm ph}$ bits imply $M = 2^{N_{\rm ph}-2}$ phase bins *per quadrant* in the procedure described above — since the range $\alpha \in [0,1]$ covers only one quadrant of phase. And $N_{\rm rf}$ bits determine $L = 2^{N_{\rm rf}-1} - 1$ *positive* levels of the rotator function; there are, of course, an equal number of symmetric negative levels, and (unless $\alpha_1 = 1$) a zero level, implicitly included in the calculation. (There seems to be no particular advantage in using levels $\pm\frac{2k+1}{2}$ which straddle zero but require an extra bit to specify.)

Results of the optimization procedure just described are summarized in Figure 1, for various cases of $N_{\rm ph}$ and $N_{\rm rf}$. For clarity of presentation, the ordinate shows $1 - |\rho_1|^2$, the fraction of power lost in unwanted harmonics, on a logarithmic scale. This, in fact, is the quantity actually minimized by ZXMIN — but more importantly, is the appropriate scaling for station-based fringe rotation where any loss occurs at both ends of each baseline. Thus the points for the 3-level rotator, $N_{\rm rf} = 2$, are all at $\sim 7.7\%$. These functions are unsatisfactory for station-based fringe rotation for this among other reasons (another is discussed later), but are shown for comparison. The general trend in Figure 1 is as expected: the approximation improves with more levels, *or* finer quantization of the transition points, but extremes of either refinement soon reach diminishing returns. The ideal balance appears to be in the range $N_{\rm ph} - N_{\rm rf} = 2 \pm 1$, implying $M/L \approx 1$, 2, or 4.

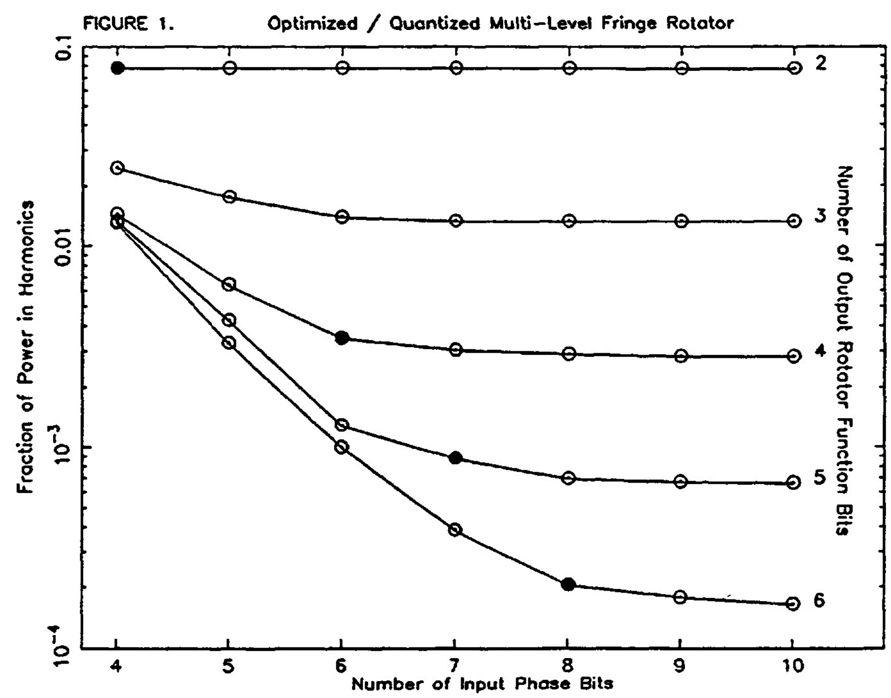


Figure 2 presents rotator functions for four representative cases with $N_{\mathrm{ph}} - N_{\mathrm{rf}} = 2$, those indicated in Figure 1 by filled circles. Each plot in the figure shows the real part of $R(t)$ over the first quadrant of phase, along with the cosine function which it should approximate. (The rest of $R(t)$ is constructed symmetrically via equations (1) and (4)). Both the optimal multi-level function (shown dashed) and the final quantized version (solid) are plotted. Ticks on the abscissa mark the quantization points.

Spectra of the same four cases appear in Figure 3, where $|\rho_n|$ (*not* its square) is plotted against $|n|$. Both axes are logarithmic, with the origin in the upper left corner, where the point plotted for ρ_1 (almost) overlaps the axes. Each tick on the ordinate represents one decade of magnitude, and each abscissa tick an *octave* of harmonic order. Since the density of harmonics increases exponentially along the abscissa, the following scheme is used to accommodate a broad range: in the first four octaves the individual points are shown by + symbols; harmonics in the second four octaves are grouped to obtain four points per octave, with the range of values being shown by the vertical line and the average by the location of the horizontal tick; this grouping is continued thereafter, but the group size does not grow beyond 16 and intervening harmonics are skipped. These plots will be discussed further at the end of the next section.

Watch the Birdie

If the loss of sensitivity due to harmonics of the fringe rotator frequency were the only consideration, Figure 1 would suggest that even the 7-level rotator function shown in Figure 2 would suffice, reducing the loss to ~0.35%. Unfortunately, harmonics can inflict worse damage than just loss of sensitivity. Coincidence of harmonics of different notes contributes "richness" in music, but in this application is better called "spurious correlation". The worst case occurs when both rotators operate at the same frequency (*i.e.*, when the baseline natural fringe rate is zero). Then *all* harmonics coincide just as the fundamentals do, and each pair produces shifted spectra which also correlate (up to the limit where the shift exceeds the bandwidth). The contribution of spurious correlation in this case is just the value $1 - |\rho_1|^2$ plotted in Figure 1. Three-level functions would have a 7.7% excess correlation, which is another reason they are unsuitable as station-based fringe rotators.

A systematic — and baseline-dependent — effect is significantly worse than a fixed-ratio sensitivity loss, even at the ~0.35% level, especially so since it may be present in some baselines for extended periods. (The station rotator frequency changes rapidly at almost all times, but two station frequencies may track for a long time.) Walker suggests in VLBA Memo 283 that such non-closing errors should be "well under 1 percent", and discusses some general considerations in VLBA Memo 388 which imply that a 45-baseline array should keep such errors to less than 10^{-4}. The 31-level rotator function shown in Figure 2 comes close to meeting this requirement.

The occurrence of this spurious correlation is entirely predictable, of course, and corrections could be applied to remove the effect. But the equal-frequency case $f_1 = f_2$ is just the worst of many such frequency pairs ($-3f_1 = 5f_2$, $3f_1 = 7f_2$, ...), which produce weaker but more numerous distortions. And we will not be helped much by the well-known

$1/n$ rolloff in harmonic strength from a square wave. Inspection of the spectra in Figure 3 shows an interesting effect of the optimization performed in defining the rotator functions. To raise the power in the fundamental, ρ_1, the procedure has systematically reduced the amplitude of the low harmonics, which otherwise make the greatest contribution to the non-fundamental power. More levels and finer quantization of the transition points both contribute to extending this flattened part of the spectrum to higher frequencies before the $1/n$ slope takes over. The result of all this is that there are a large number of low harmonics, all with roughly equal amplitude, which can be expected to produce spurious correlations too numerous to calculate and correct. Thus we will have to reduce any such effect to a harmless level, and fortunately the progressive flattening of the spectrum with increasing complexity of the rotator function should be sufficient to do this.

RECOMMENDATIONS

Referring again to Figure 1, an 8-bit phase word driving a 6-bit rotator function appears to offer adequate protection from spurious correlation effects. The phase word is easily accommodated in the current planning for fringe rotators in the FX correlator, although the 6-bit function output may be difficult in some implementations. A 5-bit function might be acceptable if necessary. Both reduce the station-based rotator sensitivity loss to insignificant levels.

The transition points for the 6- and 5-bit rotator functions are tabulated below for reference. An 8-bit phase word is assumed in both cases. The table specifies only the real part in the first quadrant, in units of 64$^{\mathrm{th}}$s of a quadrant; the two leading bits and symmetry relations required to construct the entire function are elementary, of course, but are given formally by equations (1) and (4).

$N_{\rm ph}$	$N_{\rm rf}$	α_ℓ
8	6	63 62 61 59 58 57 55 54 53 51 50 49 47 46 44 43
		41 40 38 36 35 33 31 29 27 25 23 20 17 13 8
8	5	63 60 57 55 52 49 46 43 40 37 33 29 25 20 13

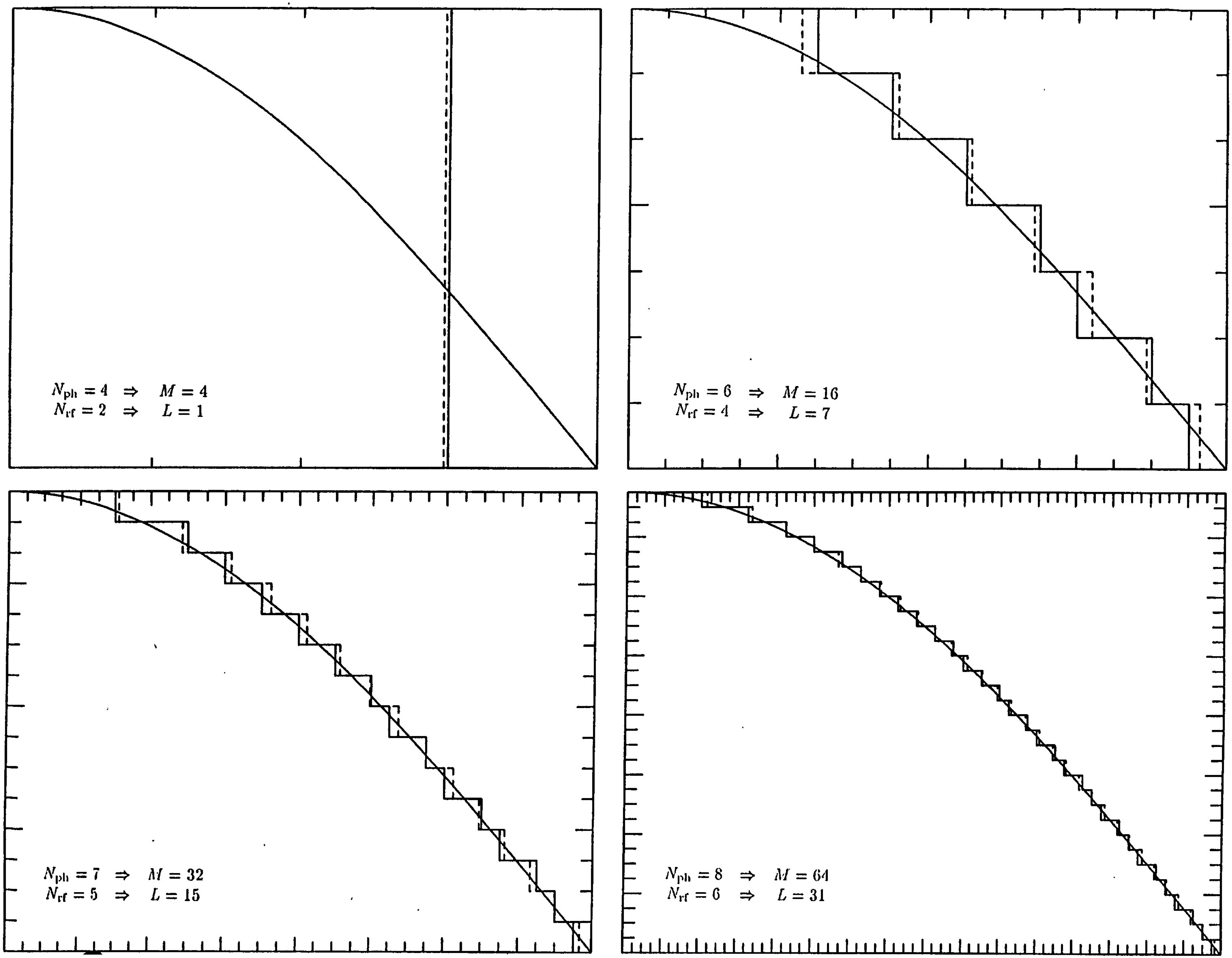
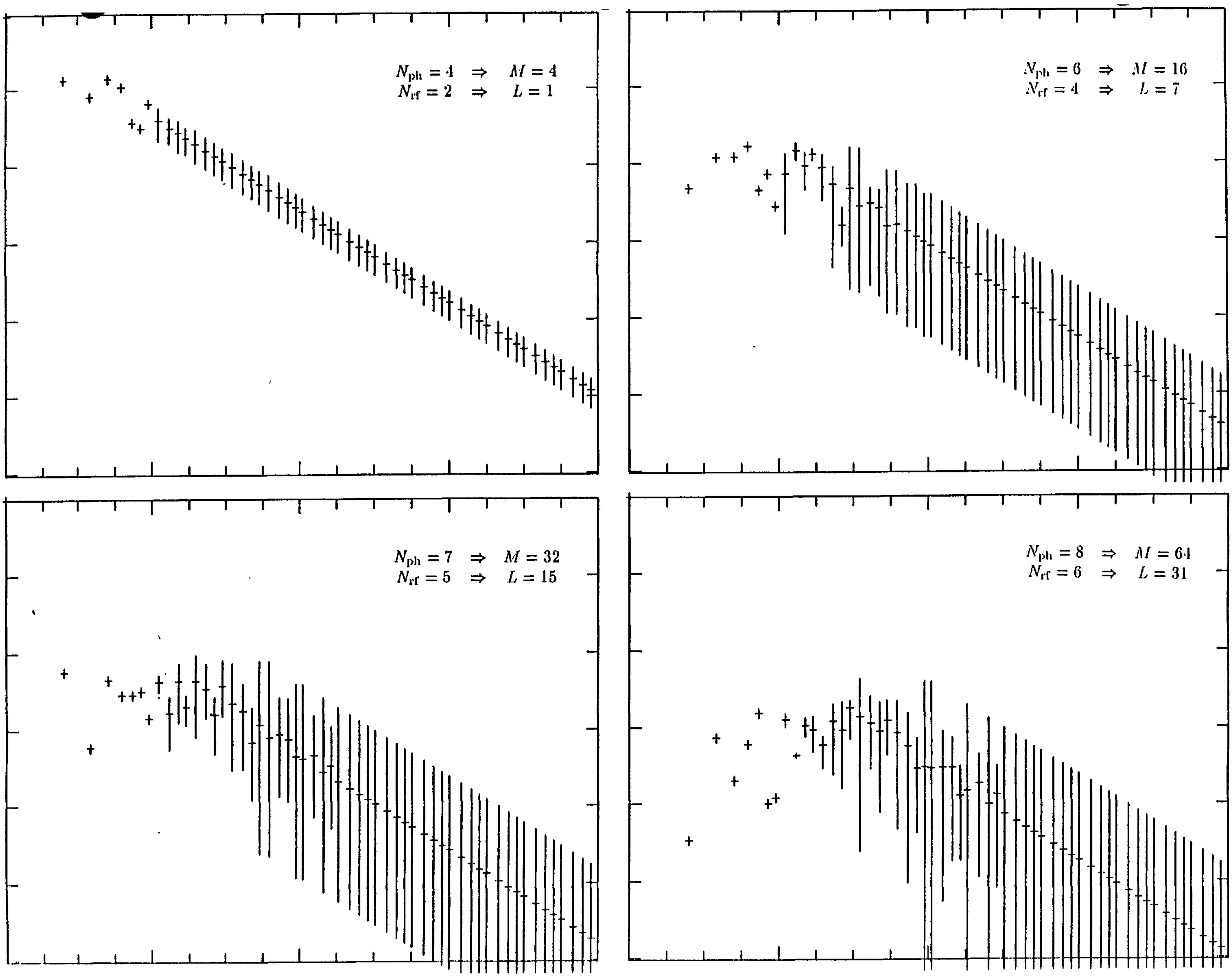

$N_{\rm ph} = 4 \;\Rightarrow\; M = 4$
$N_{\rm rf} = 2 \;\Rightarrow\; L = 1$
$N_{\rm ph} = 6 \;\Rightarrow\; M = 16$
$N_{\rm rf} = 4 \;\Rightarrow\; L = 7$
$N_{\rm ph} = 7 \;\Rightarrow\; M = 32$
$N_{\rm rf} = 5 \;\Rightarrow\; L = 15$
$N_{\rm ph} = 8 \;\Rightarrow\; M = 64$
$N_{\rm rf} = 6 \;\Rightarrow\; L = 31$

FIGURE 2



$N_{ph} = 4 \Rightarrow M = 4$
$N_{rf} = 2 \Rightarrow L = 1$
$N_{ph} = 6 \Rightarrow M = 16$
$N_{rf} = 4 \Rightarrow L = 7$
$N_{ph} = 7 \Rightarrow M = 32$
$N_{rf} = 5 \Rightarrow L = 15$
$N_{ph} = 8 \Rightarrow M = 64$
$N_{rf} = 6 \Rightarrow L = 31$

FIGURE 3