VLB ARRAY MEMO No. 105

Progress Report for the August 4th Meeting of the Configuration Group

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In this report, further results are presented on array figures of merit derived by the method outlined in VLBA Memo. No. 100. Details are worked out for quality measures comparing the density of *u-v* coverage with coverage uniform over a disk, as well as with symmetric densities having a conical profile. Also, a scheme is described for generating *u-v* sample coordinates that are uniformly spaced in *u-v* ellipse arc-length. Test results reported here include the figures of merit obtained by evaluating a few of the quality measures for some potential VLBA configurations.

Generating Sample u-v Coordinates Uniformly Spaced in u-v Ellipse Arc-Length. It was noted in Remark 3 of the concluding section of Memo. 100 that, for purposes of array quality measure comparisons, it might be desirable for the *u-v* sample coordinates to be uniformly spaced in *u-v* ellipse arc-length. The arc-length L of the segment of a *u-v* ellipse traced out during the hour-angle interval $[h_0, h_1]$ is given by

$$L = \int_{h_0}^{h_1} \sqrt{\left(\frac{du}{dh}\right)^2 + \left(\frac{dv}{dh}\right)^2}\, dh .$$

Given L and h_0, we can find h_1 by solving (using, say, Newton's method) for a root of the equation

$$q(x) = \int_{h_0}^{x} \sqrt{\left(\frac{du}{dh}\right)^2 + \left(\frac{dv}{dh}\right)^2}\, dh - L. \qquad (1)$$

To apply Newton's method, we need $q'(x)$ — which is simply equal to the integrand evaluated at x —, and we need to evaluate $q(x)$, which can be done *via* a numerical quadrature method such as Simpson's rule. Then starting with an initial guess $x^{(0)}$, which can be chosen equal to h_0, we set $x^{(k+1)} = x^{(k)} - q(x^{(k)})/q'(x^{(k)})$ and iterate until convergence. Typically only about 3 iterations are required. For more efficient computation, Equation (1) could be rewritten in terms of elliptic integrals of the second kind, in Legendre's normal form, and standard library routines then could be used to evaluate q.

A set of *u-v* samples which were generated by this scheme, and which correspond to one potential VLBA configuration, are displayed in Figure 1. Consecutive points along each *u-v* ellipse are separated by 200 km.

A Test for the Uniformity of Coverage over a Circular Disk. If the desired density of coverage in the u-v plane is given by

$$\sigma(u,v)=\begin{cases}1/\pi R^2, & u^2+v^2\le R^2,\\ 0, & \text{otherwise},\end{cases}$$

we may apply the transformations $x_1(u,v) = [\pi+\tan^{-1}(v/u)]/2\pi$ and $x_2(u,v) = \sqrt{u^2+v^2}/R$. Then $f(x_1,x_2)$, the desired density as a function of polar angle and radius, is constant as a function of x_1 and is proportional to x_2: $f(x_1,x_2) = 2x_2$. The cumulative distribution function is given by $F(x_1,x_2) = x_1x_2^2$. To evaluate the quality measure, as defined in Memo. No. 100, we need the expressions $\int_{x_2}^1\int_{x_1}^1 F(x,y)\,dx\,dy = (1-x_1^2)(1-x_2^3)/6$ and $\int_0^1\int_0^1 [F(x_1,x_2)]^2\,dx_1\,dx_2 = 1/15$. Hence, the resulting quality measure is given by the expression:

$$\begin{aligned}[T(X)]^2 = \frac{1}{N^2}\sum_{m=1}^{N}\sum_{n=1}^{N}&[1-\max(x_{1m},x_{1n})][1-\max(x_{2m},x_{2n})]\\ &-\frac{1}{3N}\sum_{n=1}^{N}(1-x_{1n}^2)(1-x_{2n}^3)+\frac{1}{15}.\end{aligned} \tag{2}$$

This is a different test than the one defined through a linear transformation of coordinates in Memo. 100, because of the nonlinearity of the transformation used here. The test given by Equation (2) here is much simpler to express and to implement. Yet another test for uniformity over a circular disk arises by defining $x_2(u,v) = (u^2+v^2)/R^2$. In both coordinates, x_1 and x_2, then, the desired density is uniform; hence one can just use the formula given in Memo. 100 for the case of uniform density over a square. Note, though, that this test differs from the one defined by Equation (2). Each of these tests is a *bona fide* test for uniformity of coverage over a disk, but no one of them is distinguished by any sort of desirable uniqueness property.

Conical Density Functions. Suppose that the desired u-v density is

$$\sigma(u,v)=\begin{cases}\frac{3}{\pi R^2(2c+1)}\left[1-(1-c)\frac{\sqrt{u^2+v^2}}{R}\right], & u^2+v^2\le R^2,\\ 0, & \text{otherwise},\end{cases}$$

where $c \equiv \frac{\text{density at } u^2+v^2=R^2}{\text{density at } u=v=0}$. Again applying the transformations $x_1(u,v) = [\pi+\tan^{-1}(v/u)]/2\pi$ and $x_2(u,v) = \sqrt{u^2+v^2}/R$, the x_1-x_2 plane density function is given by $f(x_1,x_2) = \frac{6x_2}{2c+1}[1-(1-c)x_2]$.

Omitting the intermediate details, the resulting quality measure is

$$[T(X)]^2 = \frac{1}{N^2}\sum_{m=1}^{N}\sum_{n=1}^{N}[1-\max(x_{1m},x_{1n})][1-\max(x_{2m},x_{2n})]$$
$$-\frac{1}{(2c+1)N}\sum_{n=1}^{N}(1-x_{1n}^2)(1-x_{2n}^3-(1-c)(1-x_{2n}^4)/2)$$
$$+\frac{1}{3(2c+1)^2}\left[\frac{9}{5}-2(1-c)+\frac{4(1-c)^2}{7}\right].$$

When $c = 1$, this expression reduces to Equation (2).

Results. The results of a few test runs of a program (HAZI9) implementing these array configuration quality measures are summarized below. These tests were designed only to demonstrate the potential of the method.

To check how closely the u-v samples need to be spaced (using uniform spacing in u-v ellipse arc-length) in order for the resultant quality measure to approach that which would be obtained using the arc-length definition of the quality measure alluded to in Memo. No. 100, I ran some trials tests on the D2 configuration. Setting L, the spacing of consecutive points along the u-v ellipses, to 400, 200, 100, and 50 km., and evaluating the quality measures at the eight standard declinations, I found: (1) that by $L = 200$ km. I generally had convergence to 2, and nearly 3, significant figures in $[T(X)]^2$, and (2) that for $N > 10000$ I had left only about one significant figure of accuracy in the result, because of round-off errors in the 38-bit array processor computations. (Recall that evaluation of the quality measure for 10000 sample points requires $O(10^8)$ arithmetic operations.) For $N \leq 5000$, the results, though, appear to be accurate to nearly three significant decimal digits. The quality measures computed for these tests were those for uniformity over a 15000×15000 km. square and for uniformity over a disk of radius 7500 km.

The main test results are summarized in Table 1. Here I computed quality measures for Array D7, for Array 13, for the array from Figure I-7 of Craig Walker's atlas memo, Memo. No. 97, and for Array SEG-1 (see Fig. IV-7 of Memo. 97).* Three theoretical density functions were employed: uniform density over a 20000×20000 km. square, uniform density over a disk of radius $R = 10000$ km., and a conical density function vanishing at 10000 km. ($c = 0$, $R = 10000$ km.). The separation between consecutive sample u-v points was set at $L = 200$ km. In all cases Array SEG-1, the array with two Southern Hemisphere stations,

*This array, which includes stations on Easter Island and in the Galapagos, gives extraordinarily good coverage according to most any standard that has been proposed.

was favored. Also, Array D2 was favored over Array 13 in every case. Generally Array D2 and Array I-7 were ranked about equally. Overall figures of merit can be derived by summing $[T(X)]^2$ over declination: we get (21.57, 23.40, 20.89, and 13.36, resp.)$\times 10^{-2}$ for the square, (37.30, 45.12, 37.41, and 24.95)$\times 10^{-2}$ for the disk, and (11.57, 15.78, 11.32, and 3.75)$\times 10^{-2}$ for the cone. These quality measures, of course, should only be compared in cases of identical desired density of coverage, since these tests are not distribution free; that is, the 3.75×10^{-2} for Array SEG-1 against a conical density ought not to be compared against the 13.36×10^{-2} figure for the square.

It's interesting in examining Table 1 to note, for example, that Array D2 at $\delta = 64°$ yields a total of about 1,060,000 km. of u-v tracks during the course of a 12 hour observation.

I believe that in addition to examining the standard plots of u-v coverage, as presented in Memo. No. 97, it would be very useful to see the coverage plotted in the form r-θ or r^2-θ where $r \equiv \sqrt{u^2 + v^2}$ and $\theta \equiv \tan^{-1}(v/u)$. (Note that coverage uniform over a disk in the u-v plane would be uniform in r^2-θ.) A few sample plots of this type are displayed in Figures 2–4. The figures, with their accompanying captions, are self-explanatory. Comparing Figures 3 and 4, the difference between Array SEG-1 and the U.S. arrays seems even more striking than it does when comparing u-v plots

I haven't had a chance to double check the algebra that led to the conical density function results. These results should be used only with due caution.

Table 1. (Refer to text).

		Array D2		Array 13		Array I-7		Array SEG-1	
	δ	N	$[T(X)]^2$	N	$[T(X)]^2$	N	$[T(X)]^2$	N	$[T(X)]^2$
square	64°	5304	$2.06 \cdot 10^{-2}$	5602	$2.33 \cdot 10^{-2}$	4686	$1.96 \cdot 10^{-2}$	3660	$1.65 \cdot 10^{-2}$
	44°	3542	$2.19 \cdot 10^{-2}$	3578	$2.49 \cdot 10^{-2}$	3162	$2.20 \cdot 10^{-2}$	2808	$1.67 \cdot 10^{-2}$
	30°	2640	$2.43 \cdot 10^{-2}$	2638	$2.70 \cdot 10^{-2}$	2420	$2.44 \cdot 10^{-2}$	2280	$1.67 \cdot 10^{-2}$
	18°	1954	$2.68 \cdot 10^{-2}$	1952	$2.92 \cdot 10^{-2}$	1856	$2.70 \cdot 10^{-2}$	1820	$1.71 \cdot 10^{-2}$
	6°	1382	$2.96 \cdot 10^{-2}$	1390	$3.18 \cdot 10^{-2}$	1368	$2.96 \cdot 10^{-2}$	1418	$1.78 \cdot 10^{-2}$
	-6°	1018	$3.11 \cdot 10^{-2}$	1036	$3.32 \cdot 10^{-2}$	1090	$3.04 \cdot 10^{-2}$	1220	$1.75 \cdot 10^{-2}$
	-18°	772	$3.10 \cdot 10^{-2}$	774	$3.27 \cdot 10^{-2}$	976	$2.88 \cdot 10^{-2}$	1170	$1.62 \cdot 10^{-2}$
	-30°	520	$3.04 \cdot 10^{-2}$	514	$3.19 \cdot 10^{-2}$	752	$2.71 \cdot 10^{-2}$	1010	$1.51 \cdot 10^{-2}$
disk	64°		$4.35 \cdot 10^{-2}$		$5.18 \cdot 10^{-2}$		$4.14 \cdot 10^{-2}$		$3.53 \cdot 10^{-2}$
	44°		$4.44 \cdot 10^{-2}$		$5.26 \cdot 10^{-2}$		$4.46 \cdot 10^{-2}$		$3.30 \cdot 10^{-2}$
	30°		$4.99 \cdot 10^{-2}$		$5.85 \cdot 10^{-2}$		$4.81 \cdot 10^{-2}$		$3.16 \cdot 10^{-2}$
	18°		$5.29 \cdot 10^{-2}$		$6.28 \cdot 10^{-2}$		$5.04 \cdot 10^{-2}$		$3.19 \cdot 10^{-2}$
	6°		$5.17 \cdot 10^{-2}$		$6.23 \cdot 10^{-2}$		$5.01 \cdot 10^{-2}$		$3.15 \cdot 10^{-2}$
	-6°		$4.96 \cdot 10^{-2}$		$6.16 \cdot 10^{-2}$		$4.98 \cdot 10^{-2}$		$3.14 \cdot 10^{-2}$
	-18°		$4.39 \cdot 10^{-2}$		$5.53 \cdot 10^{-2}$		$4.76 \cdot 10^{-2}$		$2.88 \cdot 10^{-2}$
	-30°		$3.71 \cdot 10^{-2}$		$4.63 \cdot 10^{-2}$		$4.21 \cdot 10^{-2}$		$2.66 \cdot 10^{-2}$
cone	64°		$1.01 \cdot 10^{-2}$		$1.48 \cdot 10^{-2}$		$9.43 \cdot 10^{-3}$		$6.31 \cdot 10^{-3}$
	44°		$1.09 \cdot 10^{-2}$		$1.56 \cdot 10^{-2}$		$1.15 \cdot 10^{-2}$		$5.26 \cdot 10^{-3}$
	30°		$1.42 \cdot 10^{-2}$		$1.94 \cdot 10^{-2}$		$1.39 \cdot 10^{-2}$		$4.71 \cdot 10^{-3}$
	18°		$1.66 \cdot 10^{-2}$		$2.25 \cdot 10^{-2}$		$1.61 \cdot 10^{-2}$		$4.71 \cdot 10^{-3}$
	6°		$1.74 \cdot 10^{-2}$		$2.34 \cdot 10^{-2}$		$1.71 \cdot 10^{-2}$		$4.98 \cdot 10^{-3}$
	-6°		$1.73 \cdot 10^{-2}$		$2.39 \cdot 10^{-2}$		$1.72 \cdot 10^{-2}$		$4.83 \cdot 10^{-3}$
	-18°		$1.55 \cdot 10^{-2}$		$2.10 \cdot 10^{-2}$		$1.55 \cdot 10^{-2}$		$3.71 \cdot 10^{-3}$
	-30°		$1.37 \cdot 10^{-2}$		$1.72 \cdot 10^{-2}$		$1.25 \cdot 10^{-2}$		$2.94 \cdot 10^{-3}$

FIGURE 1. Coverage (12 hour) for the D2 array, with sample coordinates uniformly spaced in u-v ellipse arc-length. Consecutive points along each u-v ellipse are separated by 200 km.

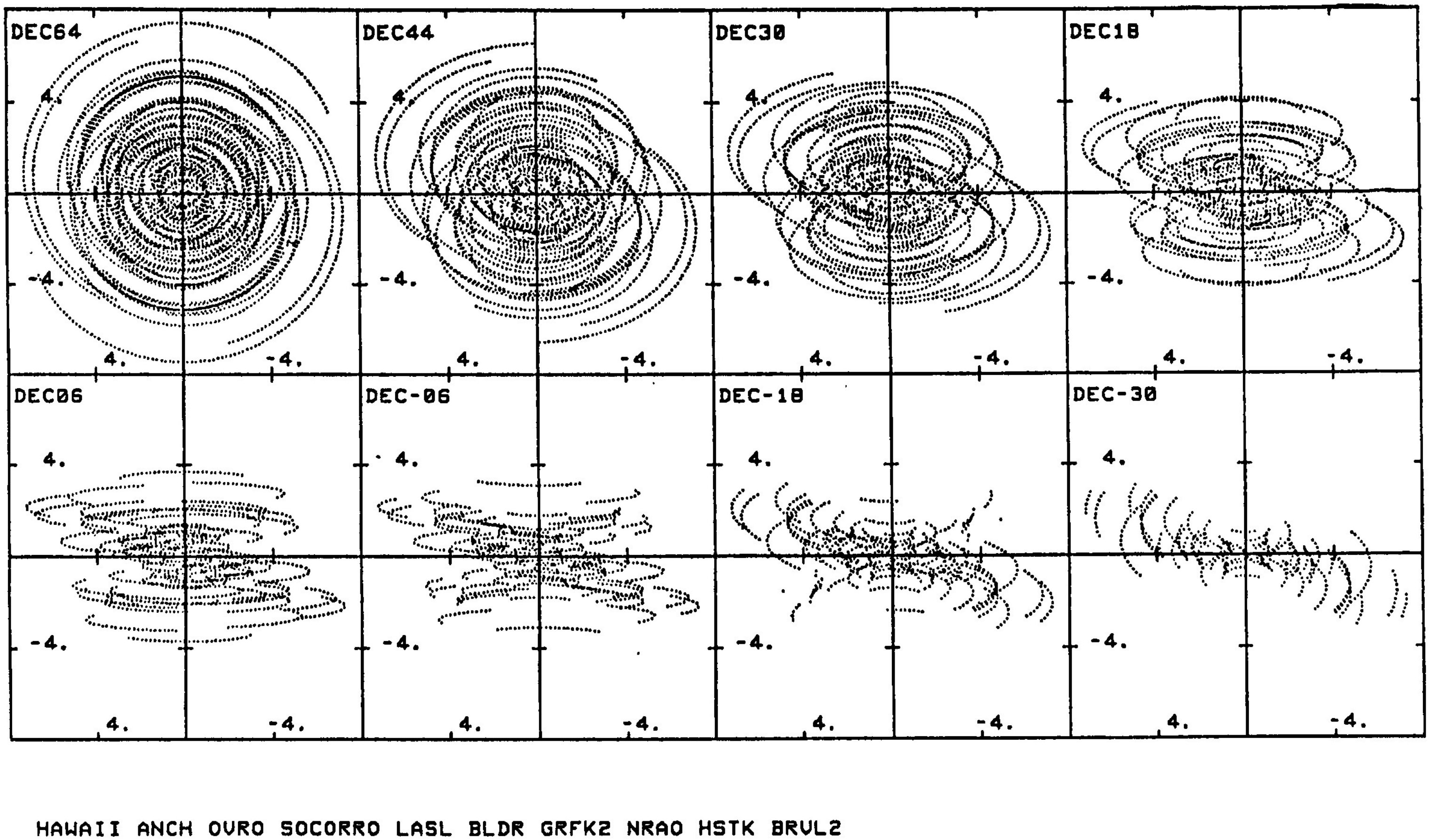


FIGURE 2. u-v coverage for the D2 array plotted in the form of polar angle $\theta = \tan^{-1}(v/u)$ *vs.* radius $r = \sqrt{u^2 + v^2}$. The axis labeling is extraneous — r ranges from 0 km. to 8000 km. and θ from $-\pi$ to π. Consecutive points on each u-v ellipse are separated by 200 km.

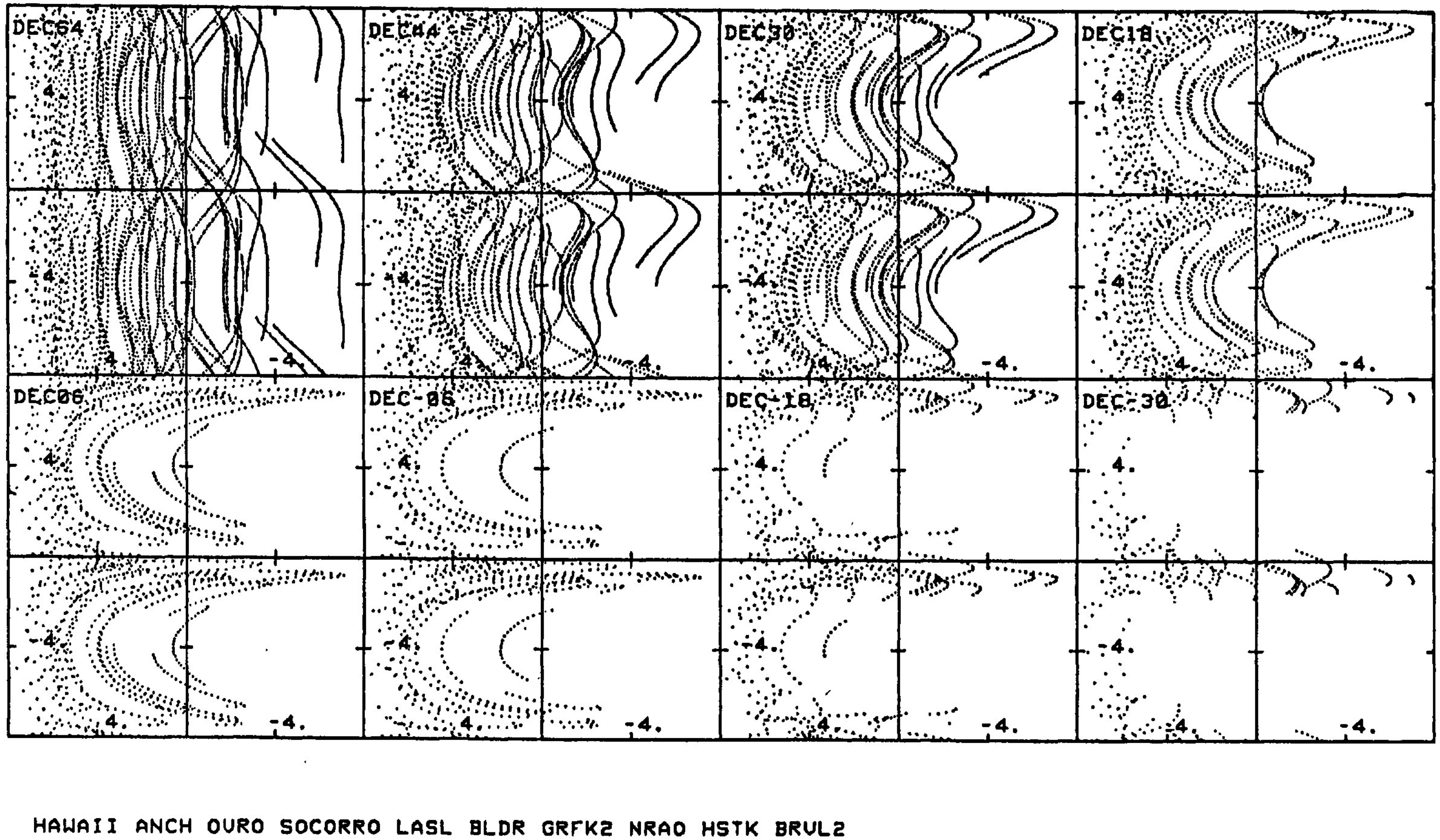
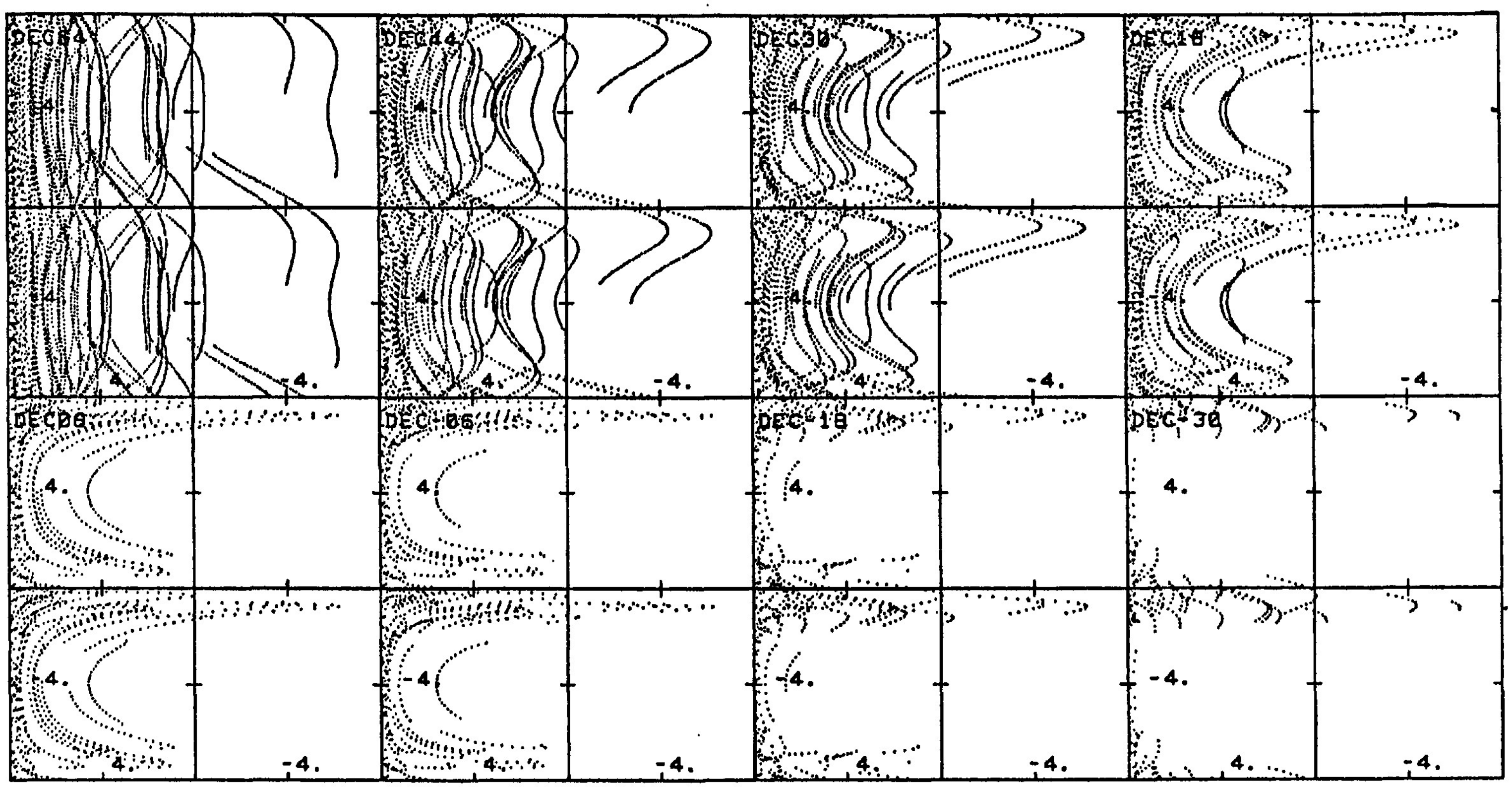


FIGURE 3. u-v coverage for the D2 array plotted in the form θ *vs.* r^2. Coverage uniform over a disk would be uniform in r^2-θ.



HAWAII ANCH OVRO SOCORRO LASL BLDR GRFK2 NRAO HSTK BRVL2

FIGURE 4. r^2-θ plots for the SEG-1 array.