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A Comment on Fringe Rotation

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We are fortunate that we have not yet been caught in and brought to shame for our use of an incorrect fringe-rotation algorithm. If we are to hold our present course, we should at least be aware of our error, keep a lookout for cases in which it might cause trouble, and be prepared to make fixes. This note discusses the faults of the present algorithm and offers a correct algorithm of essentially the same cost.

The trouble starts when we assert that multiplying a waveform by

$$\cos ut + j \sin ut = e^{jut}$$

advances the phase of the waveform by u radians/sec. In some cases, this assertion is correct, but <u>not</u> when the waveform is cos ωt:

$$\cos \omega t \; e^{jut} = 1/2[e^{-j(\omega-u)t} + e^{j(\omega+u)t}] \;.$$

Both spectral components of the waveform have been moved to the right by u radians/sec, whereas advancing the phase of cos ωt by u radians/sec means advancing the phase of the right-hand spectral component and <u>retarding</u> the phase of the left-hand component by the same amount. The algorithm does the right thing, however, when applied to the complex time function

$$e^{j\omega t} \qquad , \qquad \omega > 0 \;.$$

I hesitate to call this time function a waveform because it is not real and thus cannot be displayed as a single trace against time on an oscilloscope. In our use of the fringe-rotation algorithm, we seem always to multiply a real waveform, having conjugate positive- and negative-frequency spectral components, by e^{jut}. And thus we seem always to be in error.

An arbitrary real waveform x(t) can always be uniquely expressed as

$$x(t) = \mathrm{Re}\{X(t)\} = 1/2[X(t) + X^*(t)],$$

such that X(t) has only positive-frequency spectral components. X(t) has been called the "analytic signal" corresponding to x(t) because X(t) is analytic throughout the upper half of the complex time plane, including $\mathrm{Im}\{t\} \to \infty$. $X^*(t)$, then, has only negative-frequency spectral components. The imaginary part of X(t) is the Hilbert transform of x(t), its real part. Multiplying x(t) by e^{jut} does <u>not</u> produce a correctly phase-shifted version of x(t), but multiplying X(t) by e^{jut} <u>does</u> produce a correctly phase-shifted version of the analytic signal.

In VLBI practice, we receive a frequency-shifted version $\hat{x}(t)$ of x(t) as

$$\hat{x}(t) = \mathrm{Re}\{X(t)\, e^{jut}\},$$

and then we attempt, incorrectly, to recover x(t) as

$$\hat{x}(t)\, e^{-jut} = 1/2[X(t)\, e^{jut} + X^*(t)\, e^{-jut}]\, e^{-jut}$$

$$= 1/2[X(t) + X^*(t)\, e^{-j2ut}].$$

Provided only that the lowest-frequency spectral component of X(t) has a radian frequency greater than -2u, the above splits $\hat{x}(t)$ and $\hat{x}(t)\, e^{-jut}$ into parts containing only positive-frequency and only negative-frequency spectral components.

And for $\hat{x}(t)\, e^{-jut}$, the analytic signal part does correspond to the original $x(t)$. But the autocorrelation functions of $x(t)$ and $\hat{x}(t)\, e^{-jut}$ do not agree, nor do cross-correlation functions of $x(t)$ and $y(t)$ agree with cross-correlation functions of $\hat{x}(t)\, e^{-jut}$ and similarly constructed $\hat{y}(t)\, e^{-jvt}$.

We have, so far, avoided being caught in our mistake by one of the following:

* The positive-frequency part of the Fourier transform of $\hat{x}(t)\, e^{-jut}$ is the positive-frequency part of the (voltage) spectrum of $x(t)$. If the negative-frequency part of the transform is discarded and replaced by the conjugate of the positive-frequency part -- with the sign of the frequency reversed -- the (voltage) spectrum of $x(t)$ is correctly calculated. The squared magnitude of this spectrum is proportional to the power spectrum of $x(t)$; the product of the conjugate of this spectrum with a correspondingly generated spectrum of $y(t)$ is proportional to the cross-power spectrum of $x(t)$ and $y(t)$.

* The positive-frequency spectral components of the autocorrelation function

$$\underset{t}{\mathrm{Av}}\left\{\left(\hat{x}(t)\; e^{-jut}\right)^{*} \cdot \left(\hat{x}(t+\tau)\; e^{-ju(t+\tau)}\right)\right\}$$

agree with those of the autocorrelation function

$$\underset{t}{\mathrm{Av}}\left\{x(t)\; x(t+\tau)\right\} ,$$

so the power spectrum of $x(t)$ can be calculated by Fourier transforming the first autocorrelation function, discarding the negative-frequency part, and replacing it by the positive-frequency part, with the sign of frequency reversed. Similarly, the positive-frequency spectral components of the cross-correlation function

$$\underset{t}{\mathrm{Av}}\left\{\left(\hat{x}(t)\; e^{-jut}\right)^{*} \cdot \left(\hat{y}(t+\tau)\; e^{-jv(t+\tau)}\right)\right\}$$

agree with those of the cross-correlation function

$$\underset{t}{\mathrm{Av}}\left\{x(t)\ y(t+\tau)\right\} .$$

Since the latter is real, the cross-power spectrum of x(t) and y(t) can be calculated by Fourier transforming the first cross-correlation function, discarding the negative-frequency part, and replacing it by the conjugate of the positive-frequency part, with the sign of frequency reversed.

* In summary, we have succeeded by being careful to pay no attention to the negative-frequency spectral components produced by our incorrect fringe-rotation algorithm.

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Our fringe-rotation algorithm would evidently be correct if we were to confine its application to the analytic signal. $X(t)\ e^{jut}$ is clearly the analytic signal corresponding to x(t) advanced in phase by u radians/sec. if X(t) is the analytic signal corresponding to x(t). But samples of X(t) are complex, each requiring a real part sample and an imaginary part sample for its specification. Wouldn't this double the required number of samples per second on VLBI tapes? No, because the bandwidth occupied by x(t) is twice that occupied by its analytic signal X(t); pairs of samples would be required at half the rate. Calculating the FFT of $\hat{X}(t)\ e^{-jut}$ to obtain samples of the spectrum of the analytic signal takes less time than calculating the FFT of $\hat{x}(t)\ e^{-jut}$, because only half as many samples are involved.

The Hilbert transform of x(t) involves both the past and the future of x(t), so it is not possible to obtain the imaginary part of the analytic signal in real time by filtering its real part. One could amplitude-modulate a tone with x(t), carrier suppressed, filter out one of the sidebands, and obtain x(t) and its

Hilbert transform by phase detecting against the tone and against the tone shifted in phase by 90 degrees. A more practical scheme involves passing x(t) through a so-called 90-degree phase shifter, which delivers two allpass-filtered versions of x(t) differing in phase by 90 degrees. Although neither output is x(t), VLBI measurements will not be affected if the two phase shifts produced by the phase shifter at one station agree with what is produced at the other stations.