Detection Threshold for Global Fringe Fitting

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In Memo No. 2 of this series, "Point Source Detection Thresholds for Global Fringe Fitting", Cotton and Schwab derive a value of 2.7 for the sensitivity of the VLBA relative to a single VLBA baseline. This value is slightly more optimistic than the threshold bound given in my paper[1]. My analysis is based on the strength of noise spikes in the sum of cross-spectral functions over all baselines. While the Cotton and Schwab result is based on a phase error analysis.

Fringe search by summing over all baselines

Analogous to a fringe search on a single baseline the maximum likelihood estimate of the delay, delay rate and phases for (N-1) stations of an array on N equal elements is given by finding the rates and delays that maximize the real part of the vector sum counter rotated cross-spectral functions over time, frequency and all baselines,

i.e., find τ_i, R_i, θ_i $(1 < i \leq N)$ that maximizes

$$Re \sum_{\text{all baselines}} \sum_{\text{frequency}} \sum_{\text{time}} a_{ij}(w,t) e^{-iw(\tau_i - \tau_j)} e^{-it(R_i - R_j)} e^{-i(\theta_i - \theta_j)} \tag{1}$$

where $a_{ij}(w, t)$ = cross spectral function for accumulation at time t on baseline ij
τ_i = clock + geometrical delay to station i
R_i = clock + geometrical rate to station i
θ_i = phase to station i

assuming $\tau_1 = R_1 = \theta_1 = 0$ for the reference station. For a single baseline the real part is replaced by the magnitude making it necessary to search only over delay and rate. The phase which maximizes the real part of the sum is the phase of the sum with values of delay and rate which maximize the magnitude. With no signal present the cross-spectral functions on each baseline are uncorrelated with any other.

Upper bound on the strength of a noise spike

When a search is made over a large number of independent Rayleigh distributed random variables

$$p(Z_m) = n\, Z_m e^{-Zm^2/2} \left[1 - e^{-Zm^2/2}\right]^{n-1} \tag{2}$$

where $p(Z_m)$ is probability distribution of the maximum of n variables of unit variance.
(See Thompson, Moran and Swenson, page 264.)

For large n $\langle Zm \rangle \simeq \sqrt{2\, Log_e n}$ so that expected value for the noise peak in a search on a single baseline is increased by $\sqrt{2\, Log_e n}$ as the result of a fringe over n independent rates and delays. If the cross-spectral functions for all baselines of an N element array are added together and a search made over n values of delay and rate for the clocks of $(N-1)$ stations the noise peak in this $(N-1)$ dimensional search will be

$$\left[\frac{N(N-1)}{2}\right]^{-1/2} \left(2\ Log_e\ n^{N-1}\right)^{1/2} \tag{3}$$

where the first factor is the noise reduction due to averaging all the baselines. This simplifies to

$$\left[\frac{2}{N}\right]^{1/2} \left(2 Log_e n\right)^{1/2} \tag{4}$$

so that the noise peak is $(2/N)^{1/2}$ lower than for a single baseline or the sensitivity is enhanced by $(N/2)^{1/2}$. For 10 stations the sensitivity is increased by $\sqrt{5} = 2.236$. However the assumption that all the Rayleigh variables are independent is the worst case so 2.236 is a lower bound on the sensitivity improvement or 1/2.236 is an upper bound on the strength of a noise spike. (Also this lower bound assumes that the real part of the sum will attain a value as large as the magnitude.)

Lower bound on the strength of a noise spike

Consider the following search procedure for the purpose of evaluating the noise. First search for a maximum in the magnitude on a single baseline from station 2 to station 1. Adjust the phase of station 2 to zero the phase. Now form a partial sum of the counter rotated cross-spectra on just the baselines from station 3 to stations 1 and 2. Find a peak in the magnitude by adjusting the delay and rate of station 3 and then adjust the phase of station 3 (which is in common to both baselines) to zero the phase. Continue this process up to N stations. The noise peak for the partial sum to the k^{th} station is

$$\left[\frac{(k-1)^{1/2}}{N(N-1)/2)}\right] \left(2 Log_e n\right)^{1/2} \tag{5}$$

where the first term is the noise sigma for adding $(k-1)$ baselines and dividing by the total number of baselines. Now the values of delay, rate and phase forward using this procedure will give a sum over all baselines which is real because the phases were adjusted to make the partial sums real and will have a value equal to the sum of the magnitudes of each partial sum for the search to each added station. Thus the noise peak on the average of all baselines is

$$\frac{(2\ Log_e n)^{1/2}}{\left(N(N-1)/2\right)}\left\{1 + 2^{1/2} + 3^{1/2} + -- (N-1)^{1/2}\right\} \tag{6}$$

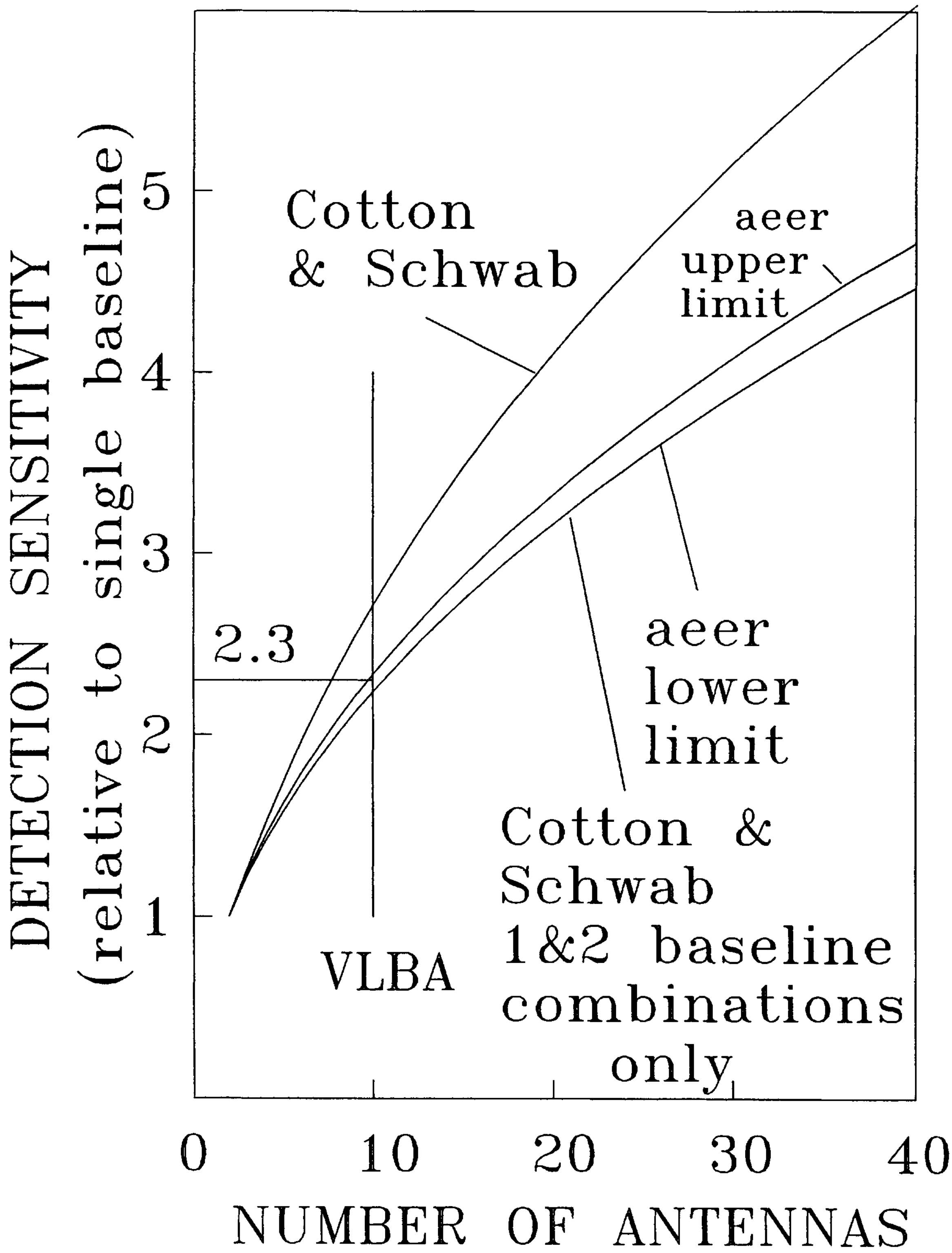
which has a value of 1/2.3309 for 10 stations. Since this represents only one prescription for finding a maximum in the average of cross-spectral functions over all baselines 2.3309 is an upper bound on the sensitivity improvement over a single baseline. This bound is below the value of 2.7 given by Cotton and Schwab but is derived by a different method making assumptions that may not be quite the same as those made by Cotton and Schwab.

I have also used a numerical matrix inversion routine to obtain the covariance matrix for the linear least squares estimation of (N-1) station phases from N(N-1)/2 phase differences which are assumed to have uncorrelated noise with unit variance on each baseline. The result is that each phase is determined with a standard deviation of $(2/N)^{1/2}$ to within a part in a thousand for all values of N tested ($2 \leq N \leq 30$). This is the same result as obtained by Cotton and Schwab for their one and two-baseline combinations.

My interpretation of this result is that global fringe fitting determines the fringe phases on a point source which an enhancement of $(N/2)^{1/2}$ over those determined by fringe fitting each baseline separately but I am not entirely sure how this relates to the detection threshold.

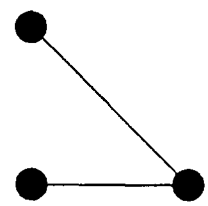
Reference

[1] A.E.E. Rogers, "VLBI fringe detection thresholds for single baselines and arrays" from Frontiers of VLBI, Edited by Hirabayashi, Inoue and Kobayashi, Universal Academy Press, Tokyo, 1991.

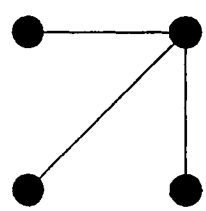


GLOBAL FRINGE SEARCH

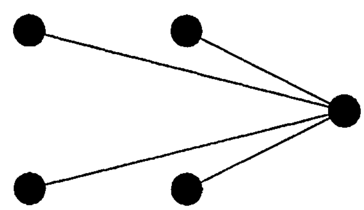
1. Search on one baseline



2. Search to third station



3. Search to fourth station



4. Search to fifth station

Sequence of searches on partial sums of baselines to estimate the noise