PUBLICATIONS OF THE NATIONAL RADIO ASTRONOMY OBSERVATORY
Volume 1 April 1961 Number 2

VERY LARGE ANTENNAS FOR THE COSMOLOGICAL PROBLEM
I. BASIC CONSIDERATIONS

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Abstract

Brightness and resolution limits are rediscussed. The atmospheric restrictions at very long and very short wavelengths are included, and special attention is given to the integrated influence of fainter background sources. At the resolution limit there should be not less than 75 beam-areas per source. The number of observable sources per steradian, N_{obs} (λ, a, b) is calculated as a function of wavelength, collecting surface and base area, for both present receivers and the best masers. The task of reaching any given N_{obs} without waste of either surface or base area defines a one-parameter family (limited at both ends) of optimum combinations of the three parameters λ, a and b.

N_{lim} is defined as the limiting number of sources per steradian which an antenna must reach in order to allow reliable distinctions between different cosmological models. Two independent estimates lead to a value of $N_{lim} = 3 \times 10^5$. The optimum solutions are calculated for this value for present receivers and the best masers. The resulting antenna dimensions are large but still within reach, e. g. a Mills cross of 2 km length and 24 m width, working at 27 cm wavelength with present receivers.

As N_{obs} gets very high, one needs only a very limited sky coverage because the number M of sources which can be handled within a reasonable time is limited. A total of M = 3000 sources would yield an accuracy of 0. 04 for the slope n of the log N, log S diagram at its upper limit and would need about two years for the survey. Including a safety factor of three, a transit instrument then would need a steerability of the beam of only 20 minutes of arc. In the case of a fixed parabolic dish with movable feed, the coma restriction then demands a ratio of focal length to diameter of at least 4. A practical realization of an antenna will be suggested in a following paper.

I. INTRODUCTION

1. Before the actual design of an antenna is started, three basic questions have to be answered:

1. What task shall the antenna achieve?
2. What properties must the antenna have in order to accomplish this task?
3. What is the cheapest way to realize these properties?

For an answer to the first question, the starting point is given by our present knowledge of radio sources; the direction in which to go is given by the challenge of unsolved problems; and the limit to which we may go is given by a compromise between the demands of these problems and a rough guess of what might be possible within a reasonable price limit. The answer to the first question will be largely a matter of personal choice

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The National Radio Astronomy Observatory is operated by the Associated Universities Inc., under contract with the National Science Foundation.

and decision.

Once the first question has been agreed on, quite a number of parts of the second question can be answered in an entirely general way, the results being valid for all possible kinds of antennas. The answer to the second question will be given by a straightforward calculation.

The third question cannot be answered in a general way. It asks for the consideration and treatment of a (principally unlimited) manifold of different types of antennas; thus no answer could be considered as being the best and final one.

2. As to the first question, we start with a discussion of whether a general-purpose or a single-purpose instrument is desired. (A single-purpose instrument will yield, if properly designed, much more information about one single problem but less information about most of the other problems, then a general-purpose instrument of equal cost.) We want to mention two arguments in favor of the single-purpose instrument. First, the possibilities of building general-purpose instruments (e.g. fully steerable, single, round dishes) seemingly have been exploited to a high degree, the largest instruments now under construction having already reached a practical price limit. Thus, within this limit, one can achieve something new for some problems only by waiving something for other problems. Secondly, provided that certain amounts of money should be spent at a number of different institutions, then the largest amount of information will be yielded if a number of different one-purpose instruments are built. Certainly, the generality of an instrument to be designed should be left large as long as possible, but should be cut down without hesitation if necessary.

As the most challenging of the presently unsolved problems one might regard the vast majority of unidentified sources: first, discrepancies between any two catalogues up to 30% are regarded as good agreement; second, we know almost nothing about the physical nature of the sources-there are a number of good reasons for believing that most of these sources are very distant extragalactic objects, but even this is not absolutely certain. Third, the number-flux density relation, for a homogenously filled Euclidean space, should yield a slope of -1.5 in the log N, log S diagram, but there is no agreement as to the observed slope, which might be somewhere between -1.5 and -2.0. Fourth, whatever this slope is, it cannot continue indefinitely towards smaller flux densities because of the limited brightness of the sky, but the lower flux limit has not yet been approached. Fifth, if the majority of the sources really are very distant objects, then in the neighborhood of this limit the effects of redshift should become appreciable, first regarding the brightness but probably the spectrum too. If an instrument could be designed which reaches and passes this limit, decisions between different cosmological models should become possible.

Only with a very large antenna can this aspect of the cosmological problem be attacked. If, furthermore, this problem would be considered as the main problem for a very large antenna, then the first question can be answered. It is the task of the antenna to reach as many sources per steradian as possible or, more precisely: to reach and pass the above mentioned limit. The term "to reach a source" should mean the following: to make sure, with a high certainty, that it is a source and neither a side lobe nor a bumpiness of the background; to give accurate positions and flux densities; to measure diameters and spectra of at least the brighter sources.

3. If we define the task as "to reach as many sources per steradian as possible", the

first question is answered, and the main part of this paper is then concerned with the second question. In this regard, the high uncertainties at the fainter limits of the present catalogues of radio sources make a rediscussion of the limitations of an antenna seem desirable. We find indeed that the resolution limit should be much more restricted than is usually assumed. It turns out that the largest number of sources per steradian will be reached at relatively short wavelengths, especially when using a maser. Therefore, one ought to include the atmospheric restrictions in the calculations of the brightness limit.

Most of the calculations to be presented here are of a general nature and are applicable to any kind of antenna. Thus the results might be of value for other purposes too.

A special suggestion for an answer to the third question will be given in a following paper.

II. ANTENNA PROPERTIES AND THE NUMBER OF OBSERVABLE SOURCES

1. The number of available sources, N_{av}.

We ask for the number of sources per steradian at wavelength λ, with flux densities ≥S, without considering any influence of the atmosphere of the earth. For this purpose, we assume the validity of two laws, which are established by observation well enough for our present estimates. First, the number of sources increases with decreasing flux density according to

$$N_{av} = \text{const } S^{-n} \tag{1}$$

where n is a constant and should be 1.5 for a homogenously filled Euclidean space (no matter what the luminosity function of the sources is), and second we assume that the spectrum of the sources is given by

$$S = \text{const } \lambda^{x} \tag{2}$$

where x is another constant. Because the observations do not agree as to the value of n, which might be between 1.5 and 2.0, we leave n open as long as possible. As to the value of x, it seems to be possible to adopt without too much objection a fixed value of

$$x = 0.8 \tag{3}$$

for the majority of the unidentified sources in which we are interested (Whitfield, 1959). Combining the two laws, we adjust the constant for a best fit with the 3C-Catalogue (Edge et al. 1959) at an average flux density

$$N_{3C} = 15 \text{ sources/steradian at} \begin{cases} S = 20 \times 10^{-26} \text{ W m}^{-2}\text{cps}^{-1} \\ \lambda = 1.89 \text{ m} \end{cases} \tag{4}$$

and obtain

$$N_{av}(S,\lambda) = 15\left(1.20 \times 10^{-25} \frac{\lambda^{0.8}}{S}\right)^{n} \tag{5}$$

where S is measured in W $m^{-2}cps^{-1}$, λ in m, and N_{av} in sources per steradian.

2. The number of visible sources, N_{vis}

After the number of available sources above a given flux density at a given wavelength has been expressed by (5), we ask for the number of sources at a given wavelength which are visible with an antenna having a given circular collecting surface of diameter a, without regard to resolution. Thus N_{vis} is the brightness limit at wavelength λ according to the collecting surface of an antenna. We call:

T_g = Temperature of galactic plus extragalactic background radiation
T_a = Temperature of atmospheric radiation
T_n = system noise temperature
$T_o = T_g + T_a + T_n$ = overall noise temperature
T_s = smallest measurable antenna temperature difference (source)
B = bandwidth of receiver
f = frequency of receiver $p = B/f$ (6)
t = observing time
q_n = signal to noise ratio, according to T_o

The rms noise fluctuation, in temperature units, of a Dicke-type receiver is given by

$$0.7\, T_o(Bt)^{-1/2}$$

and if we demand a certain minimum signal to noise ratio, q_n, we have

$$T_s = 0.7\, q_n T_o (Bt)^{-1/2} .$$

If we regard the collecting surface as being a single round dish of diameter a, and adopt the effective surface as 0.7 of the geometrical one, $A = 0.7\pi(a/2)^2 = 0.55\, a^2$, and using the relation $T = SA/2k$, we get the following expression for the smallest measurable flux density of a source:

$$S = 3.51 \times 10^{-23} \frac{q_n T_o}{a^2\sqrt{Bt}} \quad . \tag{7}$$

In order to obtain a lower limit of q_n we have made a series of experiments with random numbers, simulating three observing points per beamwidth with different q_n and asking for the relative errors of flux density (Δ S/S) and of position and diameter (error/beamwidth). As one should expect, the products of these relative errors, multiplied by q_n, stay constant and are about equal to one, but this holds only if $q_n \geq 10$. The products increase by 25% at $q_n = 5$ and by 100% at $q_n = 3$. Furthermore, the fraction of complete misinterpretations of sources is 1/20 at $q_n = 5$ and 1/4 at $q_n = 3$. We thus suggest regarding

$$q_n = 5 \tag{8}$$

as a reasonable lower limit for the signal to noise ratio.

Because our task is to observe a very large number of sources, we choose a relatively short time of observation, t = 10 sec. For the bandwidth we adopt 5% of the frequency for all frequencies, $p = 0.05$. Measuring λ and a in meters, we then have:

$$S = 1.43 \times 10^{-26} \frac{T_o\sqrt{\lambda}}{a^2} . \quad (9)$$

The above formulae would be valid if there were no limitations on the observable spectral range of wavelengths. One such limitation is the absorption of the Earth's atmosphere, which is negligible for medium wavelengths, becomes appreciable for wavelengths below 3 cm and above 10 m, and cuts off at about 0.8 cm and 60 m. In practice, however, the observable range is still more restricted at both ends. Man-made interference makes the observations more and more difficult above 3 m, and almost terminates rountine observations at about 30 m. The limitation at the shorter wavelengths is given by the slow scintillations of the sources and the slow variations of the atmospheric radiation, beginning below 10 cm and terminating useful observations at about 1 cm.

The observable spectrum thus is limited by man-made noise on one side and by slow atmospheric variations on the other side. Both limitations have about the same effect as an increase of the noise level, T_o. Therefore, we define a limiting function $g(\lambda)$ by which this increase is described, with $g = 1$ for medium wavelengths and $g = 0$ at both ends:

$$g(\lambda) = \frac{T_o}{\text{increased (effective) noise temperature}} . \quad (10)$$

The smallest measurable flux density of a source then is:

$$S(a, \lambda) = \frac{\phi(\lambda)}{a^2} \quad (11)$$

with

$$\phi(\lambda) = 3.51 \times 10^{-23} \frac{q_n T_o}{g\sqrt{Bt}} = 1.43 \times 10^{-26} \frac{T_o\sqrt{\lambda}}{g} . \quad (12)$$

The numerical values adopted are given in Table 1. The galactic plus extragalactic radiation, T_g, is taken from Kraus and Ko (1957); the radiation of the Earth's atmosphere, T_a, is taken from Van Vleck (1951); the system noise temperatures, T_n, are from F. Drake (1961). Most difficult to estimate is the limiting function $g(\lambda)$ where no usable figures seem to be available. The values adopted in Table 1 are the result of discussions with several observers and should be regarded as a mere rough guess for short observing times, in order to account at least to some extent for the actual limitations.

The following calculations are carried out for two cases: the present types of vacuum tube receivers usually used, and masers. The system noise temperatures for present receivers are given in column 5 of Table 1, and for a good maser of the future we have adopted a constant value of $T_n = 20° K$. Using these values, Table 2 shows the smallest measurable flux densities according to (11) and (12).

Finally, we have to insert the smallest flux densities, equation (11), into the number of available sources, equation (5), in order to get the number of visible sources, N_{vis}:

$$N_{vis}(a,\lambda) = 15\left(\frac{1.20 \times 10^{-25}\,\lambda^{0.8}}{\phi\,(\lambda)}\;a^2\right)^n = \psi(\lambda).\,a^{2n}, \qquad (13)$$

$$\psi(\lambda) = 364\left(\frac{\lambda^{0.3}}{T_o}\right)^{1.5}, \text{ for } n = 1.5.$$

TABLE 1

Adopted values for noise temperatures, defined in (6), and for the limiting function, defined in (10).

f	λ	T_g	T_a	T_n	T_o		$g(\lambda)$
M cps	m	Degrees Kelvin			present	maser	
10	30	200 000			200	000	.02
15	20	73 000			73	000	.14
20	15	36 000			36	000	.40
30	10	12 800		180	12	900	.75
100	3	630		250	880	650	1.00
300	1	41		350	390	61	1.00
1 000	.3	2.0	3.8	540	546	26	1.00
3 000	.1	.12	4.0	900	904	24	1.00
10 000	.03		8.5	1 650	1 660	28	.75
15 000	.02		16	2 300	2 320	36	.40
20 000	.015		40	3 000	3 040	60	.14
25 000	.012		110	3 900	4 010	130	.05
30 000	.010		90	5 000	5 090	110	.02

At this point, a numerical value for n must be adopted. The homogeneously filled Euclidean space would yield n = 1.5. The 3C-Survey (Edge et al. 1959) gave n = 2.0, but the authors remark that the actual value might be somewhat lower if some of the brighter sources were missed because of their larger diameters. According to Mills and Slee (1957), their observed value of about 1.7 goes down to almost 1.5 by the application of two corrections connected with the resolving power. Thus, the most conservative value of 1.5 still might be the best guess. Therefore, we shall use

$$n = 1.5 \qquad (14)$$

for the following calculations, without necessarily regarding this as the true value of n. The values of N_{vis} in Table 2 have been calculated with n = 1.5 and are shown in figure 1.

3. The "noise" of background sources.

After having calculated the brightness limit, we next have to treat the resolution limit of an antenna. In earlier investigations it was only demanded that the antenna must be able to separate the observed sources from their neighbors, but Mills and Slee (1957) and others have drawn attention to the fact that the numerous faint background sources will restrict the resolution limit considerably more. In our opinion, this restriction is still higher than in the previous estimates.

TABLE 2

Smallest neasurable flux densities according to (11) and
Number of visible sources according to (13) with n = 1.5

f	$\phi = S \quad a^2$		$\psi = N_{vis} \quad a^3$	
M cps	present	maser	present	maser
10	7.9×10^{-19}		6.5×10^{-8}	
15	3.3×10^{-20}		3.7×10^{-6}	
20	5.0×10^{-21}		4.5×10^{-5}	
30	7.8×10^{-22}		4.5×10^{-4}	
100	2.2×10^{-23}	1.6×10^{-23}	2.3×10^{-2}	3.6×10^{-2}
300	5.6×10^{-24}	8.7×10^{-25}	4.7×10^{-2}	7.7×10^{-1}
1000	4.3×10^{-24}	2.0×10^{-25}	1.7×10^{-2}	1.6
3000	4.1×10^{-24}	1.1×10^{-25}	4.7×10^{-3}	1.1
10 000	5.5×10^{-24}	9.3×10^{-26}	7.1×10^{-4}	3.2×10^{-1}
15 000	1.2×10^{-23}	1.8×10^{-25}	1.4×10^{-4}	7.3×10^{-2}
20 000	3.8×10^{-23}	7.5×10^{-25}	1.7×10^{-5}	6.2×10^{-3}
25 000	1.3×10^{-22}	4.1×10^{-24}	2.1×10^{-6}	3.7×10^{-4}
30 000	3.7×10^{-22}	7.9×10^{-24}	3.5×10^{-7}	1.0×10^{-4}

Units: S in W $m^{-2} cps^{-1}$; a in m; N in sources/steradian

Suppose we want to observe sources with flux densities $\geq S_{obs}$ and there are μ beam solid angles per source of this flux density. The beam then contains in the average the number

$$\frac{1}{\mu}\left(\frac{S_{obs}}{S}\right)^n$$

of sources with flux densities $\geq$ S; and the average number of sources with flux densities S...S+dS within the beam is

$$dN = \frac{n}{\mu}\left(\frac{S_{obs}}{S}\right)^n \frac{dS}{S}$$

This number will follow a Poisson distribution (no clustering assumed) with a standard deviation of $\sqrt{dN}$, the standard deviation of its contribution to the background thus being $S\sqrt{dN}$. These deviations add up quadratically, and the total standard deviation of the background then is:

$$\Delta S = \left(\int S^2 dN\right)^{1/2} = \left(\frac{n\, S_{obs}^n}{\mu} \int_{S_{lim}}^{S_{obs}} \frac{dS}{S^{n-1}}\right)^{1/2} \qquad (15)$$

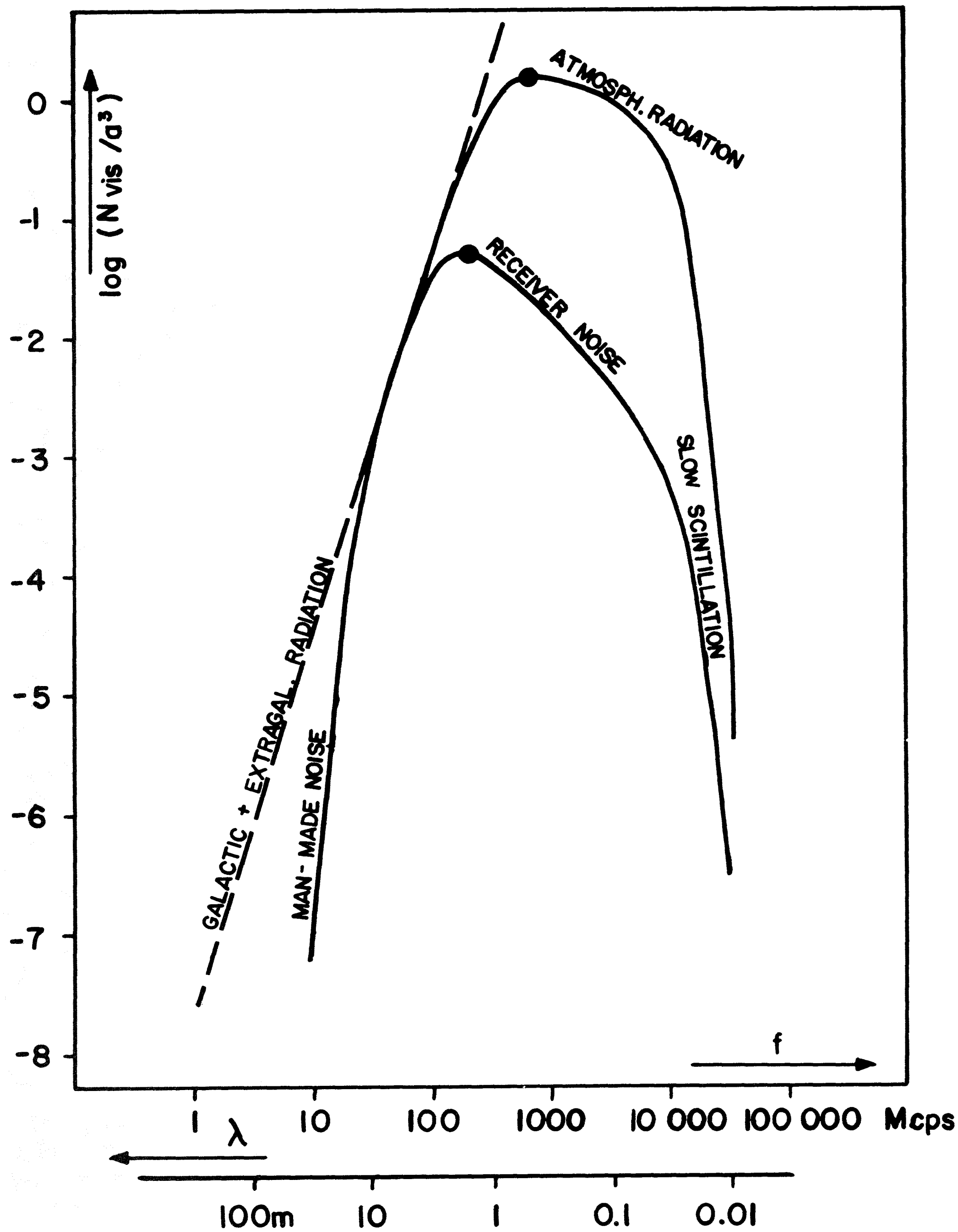

Fig. 1. The brightness limit. N_{vis} (a, λ) = number of visible sources per steradian. A = antenna diameter in meters. Upper curve: maser with Tm = 20° K; maximum at λ = 0.36 m. Lower curve; present receivers; maximum at λ = 1.35 m.

This formula gives the exact value of ΔS, if the slope n of the log N/log S plot stays constant to a lower limit S_{lim}, and if there are no sources fainter than S_{lim}. But as long as $S_{obs} >> S_{lim}$, formula (15) should be a good approximation. We call

$$q_b = \frac{\text{Signal}}{\text{noise}} = \frac{S_{obs}}{\Delta S} \tag{16}$$

and the integration of (15) then yields

$$q = \frac{\sqrt{\mu}}{Q} \quad \text{with } Q = \begin{cases} \left(\frac{n}{2-n}\left[1-\left(\frac{S_{lim}}{S_{obs}}\right)^{2-n}\right]\right)^{1/2} & \text{for } n < 2 \\ \left(2 \ln\left(\frac{S_{obs}}{S_{lim}}\right)\right)^{1/2} & \text{for } n = 2 \end{cases} \tag{17}$$

TABLE 3

The quantity Q, as defined in (17)

n < 2

n	$S_{obs} >> S_{lim}$	$S_{obs} = 10\ S_{lim}$
1.5	1.73	1.43
1.7	2.38	1.68
1.8	3.00	1.82

n = 2

S_{obs}/S_{lim}	Q
10	2.14
100	3.03
1000	3.72

Some values of Q have been calculated and are shown in Table 3 in order to illustrate the range of variation of Q according to different assumptions. Fortunately, this range is rather limited. We take, as before, n = 1.5 and, to be on the safe side for the stronger sources, we adopt for the following calculations the value Q = 1.73. Equation (17) then reads:

$$q_b = \frac{\sqrt{\mu}}{1.73} \tag{18}$$

This means that when we observe sources of a given flux density, and there are μ beam solid angles per source of this or higher flux density, then the signal to noise ratio due to the background sources is given by equation (18). We want to emphasize that there is no way of increasing this ratio once the size of the antenna and the wavelength are chosen, whereas the signal to noise ratio due to receiver noise and continous galactic radiation still can be increased by an increase of observing time or bandwidth, or by repeated observations of a source. Considering $q_b = 5$ as a proper lower limit (see text to formula 8) there should be not less than

$$\mu = 75 \text{ beam solid angles per source.} \tag{19}$$

By some experiments with random numbers we found that the resulting uncertainty of position measurements as well as of diameter measurements (error/beamwidth for both cases) is about equal to the uncertainty of the flux density measurement, given by q_b. Thus, position and diameter accuracies, too, are limited by (18) and cannot be increased in any way for a given antenna size and wavelength.

4. The number of resolvable sources, N_{res}.

According to its definition $\mu = 1/N\alpha$, where α is the solid angle of the beam. Using $\alpha = \pi\,(\beta/2)^2$ and $\beta = 1.2\ \lambda/a$ for the beamwidth β, we get

$$\mu = \frac{0.885}{N}\left(\frac{a}{\lambda}\right)^2 \tag{20}$$

and finally

$$N_{res} = \frac{0.295}{q_b^2}\left(\frac{a}{\lambda}\right)^2 \tag{21}$$

for the number of sources per steradian resolvable with a signal to noise ratio q_b, for antenna diameter a and wavelength λ (both measured in the same units).

In the case of a complex antenna system, equation (21) needs modification. The quantity a was defined in section II, 2 as the diameter of the collecting surface of the antenna in case of a single round dish, and was used for the calculation of the brightness limit. In order to increase the resolution without changing the brightness limit, the same collecting surface can be spread over a larger base area of diameter b. In this case we may replace a by b in (21) if one condition is fulfilled:

> The effective beam solid angle of the system must be the same as for a single round dish of diameter b. (22)

This condition could be fulfilled, for example, by spreading a large number of small antenna pieces randomly over the base area b, by aperture synthenis, or by using a Mills cross of arm length b, where the small area of the pencil beam is reached by phase switching between the two arms (in phase and out of phase). If the condition is not fulfilled, as in the case of a fixed two-antenna interferometer, the effective total beam solid angle has to be used, including all strong sidelobes, which decreases the resolution limit considerably. In the following, we assume that condition (22) is fulfilled.

As to the minimum value of q_b, we should take the same value as we did for the usual signal to noise ratio in (8):

$$q_b = 5 \tag{23}$$

which gives

$$N_{res}\,(b,\lambda) = 1.18 \times 10^{-2}\left(\frac{b}{\lambda}\right)^2 \tag{24}$$

The resolution limit given in (24) is considerably more restricted than is usually assumed, but then the agreement of the fainter sources between any two catalogues is not too good, either. We even think that our adopted value of $q = 5$ for (8) and (23) still might be too low. By a thorough Monte Carlo treatment, this question could be settled.

5. The number of observable sources, N_{obs}.

By combining the brightness limit, N_{vis}, and the resolution limit, N_{res}, we get the number of observable sources per steradian as the minimum of both:

$$N_{obs}(a, b, \lambda) = \text{Min} \left[N_{vis}(a, \lambda), \ N_{res}(b, \lambda) \right] \tag{25}$$

The results are shown in figure 2 for present receivers and in figure 3 for a good maser of the future. A similar picture was given by Kraus (1958), with less details and using less restricted assumptions. For the sake of clearness we summarize all assumptions included in the numerical values of figures 2 and 3. First, a number-flux density law according to (1) and a spectral law according to (2) are assumed. For the exponents we adopted $x = 0.8$ in (3) and $n = 1.5$ in (14), and the constants were fitted to the 3C catalogue for medium flux densities. Second, we adopted an observing time of $t = 10$ sec and a bandwidth of 5% of the frequency. Third, the adopted noise temperatures and the limiting function are given in Table 1, with a maser temperature of 20° K. Fourth, as a lower limit for the two signal to noise ratios of actual noise and of background sources we chose $q_n = q_b = 5$ in (8) and (23). Fifth, the effective beam solid angle of an antenna system is supposed to be the same as for a single dish of diameter b, according to (22).

Law (11) cannot continue to very high values of N with a constant slope (see section III, 1). Above about $N = 10^5$ the scales of figures 2 and 3 should be elongated considerably, but we cannot tell how much and therefore leave the scales unchanged. The curved lines in figures 2 and 3 are the brightness limits; the straight lines, the resolution limits; and the heavily marked crossing points (where a = b) are for a single dish antenna. From these figures we can read the number of observable sources per steradian for any combination of collecting surface, base area and wavelength. It should be mentioned that N_{obs} has to be decreased by about a factor of 2 in the vicinity of the crossing point of a given resolution limit with a given brightness limit, because then the receiver noise and the "background noise" add up.

For example: with a single dish of 25 m diameter and with a present receiver, a maximum number of $N_{obs} = 85$ sources per steradian could be observed at a wavelength of 22 cm. By spreading the same antenna surface over a base area of 800 m diameter and observing at $\lambda = 1.35$ m, the maximum number increases to 800. If we use a maser of 20° K noise temperature, we get $N_{obs} = 3000$ at $\lambda = 3.5$ cm for the same single dish, and a maximum of $N_{obs} = 25000$ at $\lambda = 36$ cm for the same surface spread over a base area of 1.5 km diameter. The variation of N_{obs} in this example is 2-1/2 powers of ten, using the same collecting surface in different ways, but always at the most effective wavelength.

The single dish has its maximum efficiency at the heavily marked crossing points which lie at relatively short wavelengths; and the larger the dish, the shorter is this optimal wavelength as shown in Table 4. For a dish of 100 m diameter, for example, we could reach an optimum of $N_{obs} = 3200$ at $\lambda = 14$ cm, but if we want to observe at $\lambda = 2$ m, the resolution limit cuts down to only $N_{obs} = 50$. This same number at the same wavelength could be observed with a much smaller collecting surface of a = 13 m,

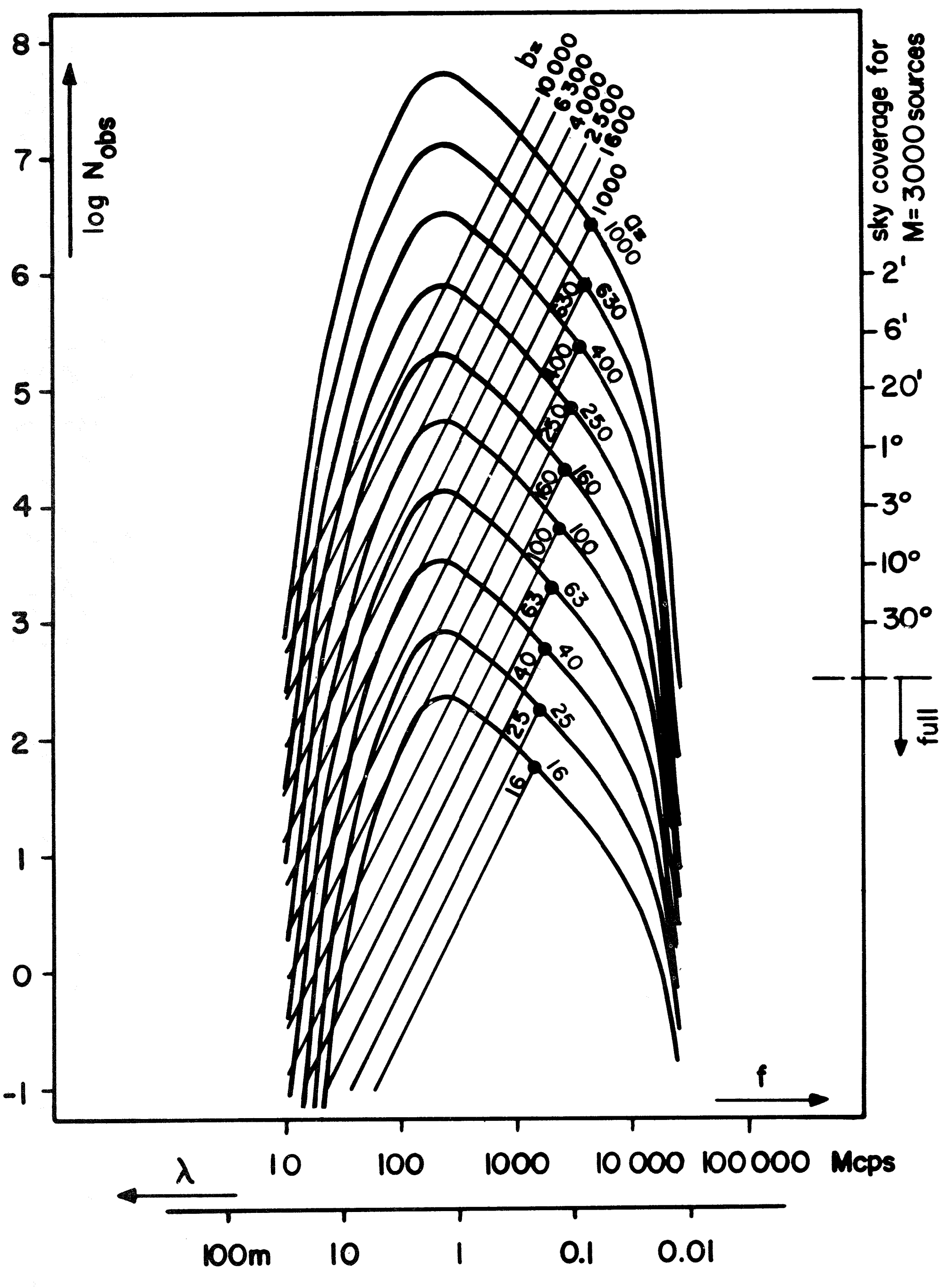

Fig. 2. The number of observable sources per steradian for present receivers. A = diameter of round dish (in m) with surface equal to the total collecting surface of the antenna. B = diameter of base area (in m) over which the antenna system is spread.

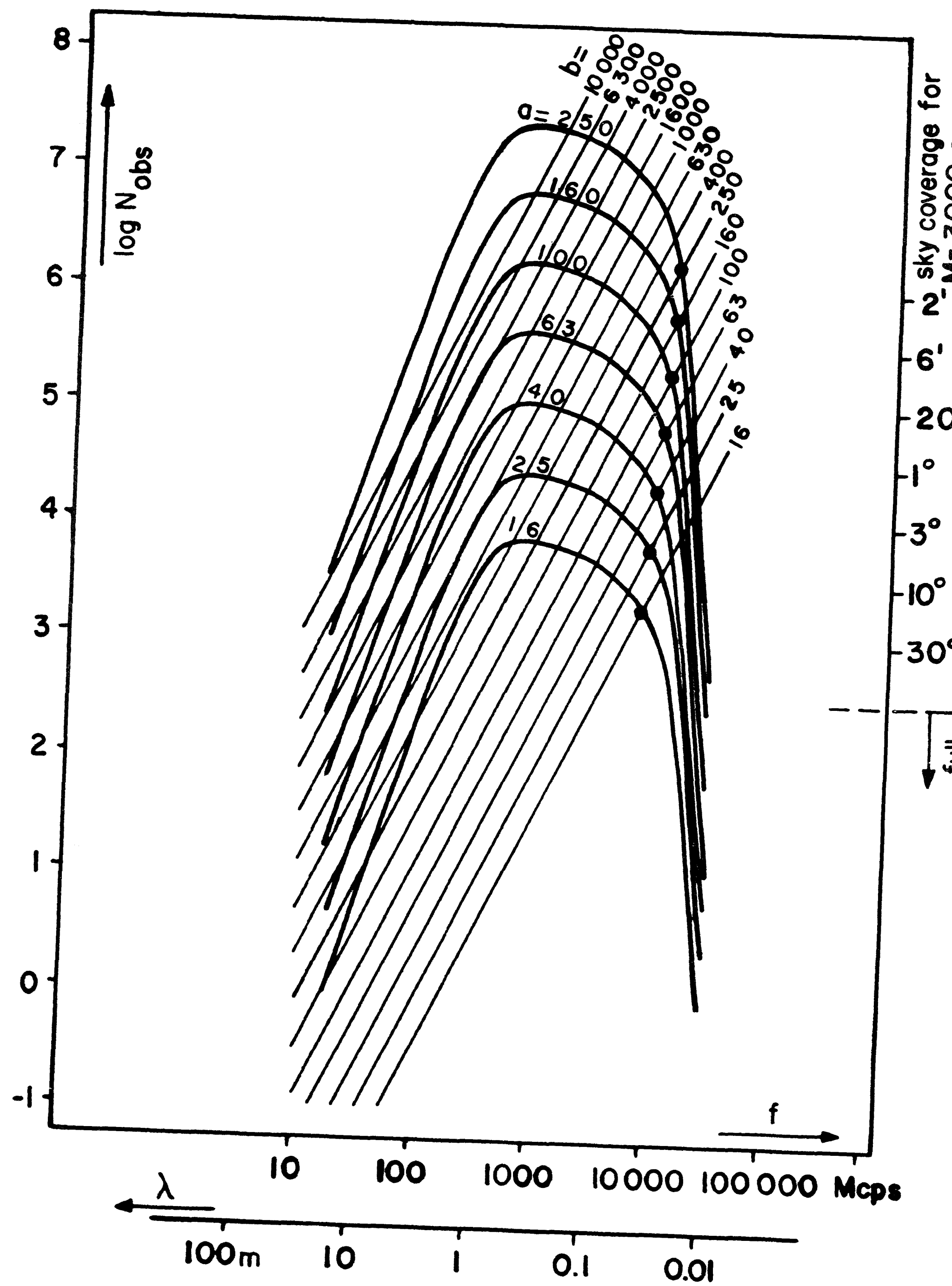

Fig. 3. The number of observable sources per steradian, for maser with Tm = 20° K.

spread over these same 100 m. This means that, if we observe at 2 m with a 100 m dish, then 98% of the antenna surface is just wasted. On the other hand, in the case of a base b much larger than a, we have two additional problems: how to distribute the antenna parts in order to get a minimum beam solid angle, and more serious, how to collect all of the information to one point over distances of at least b/2. In spite of the above mentioned "waste of surface", the single dish still has the convenient advantage of reflecting all information to one feed, in the proper phase for all wavelengths.

TABLE 4

Maximum efficiency of single round dish antenna.
a = diameter of antenna
λ = wavelength, where N_{obs} = maximum
s = sky cover needed for total of 3000 sources with transit instrument.
N_{obs} = number of observable sources per steradian

a m	present receivers λ cm	 N_{obs} sterd^{-1}	 sky cover s	maser 20° λ cm	 N_{obs} sterd^{-1}	 sky cover
16	24	26	-	4.0	1 000	35°
25	22	85	-	3.5	3 000	12°
40	19	290	-	3.1	10 000	3°.5
63	16	930	37°	2.8	30 000	1°.2
100	14	3 200	11°	2.6	91 000	23'
160	12	10 500	3°.3	2.4	270 000	8'
250	10	34 000	1°.0	2.2	750 000	3'

6. The Sky Coverage

Our main concern is to reach a very large number of sources per steradian, but only a limited total number is needed for solving a problem, and only a limited number of some thousand sources can be handled within a reasonable time. Thus we have to restrict the observations to selected areas, and this is a very convenient restriction. If we consider a transit instrument where the sky moves through the beam in east-west direction, we need to move the beam by only a very small amount in the north-south direction in order to get a sky coverage large enough for a manageable number of sources, and a strip around the whole sky should yield a fairly representative average.

If N_{obs} is the number of observable sources per steradian which we want to reach with the antenna, and M is the total of sources which we want to handle, and if the observations are to be made around declination δ, and the declination coverage (the movability of the beam in declination), s, is small compared to 180°, we then have

$$s = \frac{547}{\cos\delta}\frac{M}{N_{obs}} \text{ minutes of arc,} \tag{26}$$

and with $\delta = +38°$ (zenith at Green Bank),

$$s = 695 \frac{M}{N_{obs}} \text{ minutes of arc.} \tag{27}$$

In order to decide as to the value of M we have to compromise between two demands: we want to observe the slope n with a good accuracy (M high) but we don't want to spend too many years at this (M low). It can be shown that Δ n, the probable error of a measured slope n, is connected with M by

$$\frac{\Delta n}{n} = \frac{1.44}{\sqrt{M}} \tag{28}$$

If we take ν observing points per beamwidth in each scan and also ν scans per beamwidth interval in declination, and a time t per observing point (integration time), then t_s, the observing time needed for the whole survey, is given by

$$t_{obs} = \nu^2 \mu M t \tag{29}$$

We take t = 10 sec (see text to formula 9), $\mu = 75$ according to (19), and regard $\nu = 3$ as sufficient. In order to be on the safe side, we count on only 1/3 of useful observing time and finally get for the total time of the survey

$$t_{total} = 3 t_{obs} = 6.4 \times 10^{-4} M \text{ years} \tag{30}$$

The compromise we want to recommend is

$$M = 3000 \quad \text{with} \quad \left[\begin{array}{l} \Delta n = 0.04 \\ t_{total} = 2 \text{ years} \end{array} \right. \tag{31}$$

and this survey would need a declination coverage of

$$s = \frac{2.08 \times 10^6}{N_{obs}} \text{ minutes of arc} \tag{32}$$

For comparsion with (32) we calculate how small a beamwidth β is needed for reaching a given N_{obs}. The resolution limit was defined in chapter 3 by demanding that $\mu = 75$. By definition, $\mu = 1/(N_{res}\, \alpha)$ with $\alpha = \pi\beta^2/4$. The most efficient point to work at is the crossing of resolution limit and brightness limit. At this point we have $N_{res} = 2 N_{obs}$ because of the addition of receiver noise and background mentioned in the previous chapter. We thus finally get

$$\beta = \left(\frac{2}{75\pi N_{obs}} \right)^{1/2} \text{ radian} = \frac{317}{\sqrt{N_{obs}}} \text{ minutes of arc} \tag{33}$$

7. Coma and focal ratio

We need only very small sky coverage, and therefore one of the possible solutions might be a fixed parabolic surface with a movable feed. The limitation in sky coverage then is given by coma. We call:

c = length of coma, in minutes of arc
F = focal length ⎤ in same units
a = antenna diameter ⎦
v = F/a = focal ratio

According to Schwarzschild (1905) we have

$$c = 0.094 \frac{s}{v^2} \tag{34}$$

where the needed sky coverage, s, again is measured in minutes of arc. If we impose the condition that the coma should be smaller than 1/5 of the beamwidth β,

$$c \leq \frac{1}{5} \beta \tag{35}$$

and using the formulae (32) and (33), we arrive at:

$$v \geq \frac{55.5}{N_{obs}^{1/4}} \tag{36}$$

This means: when an antenna is designed so as to reach a maximum of N_{obs} sources per steradian, and the sky coverage is fixed so as to yield a total of 3000 sources and is achieved by moving the feed only, then the focal ratio must have a minimum value given by (36) in order not to violate the safety limit set by (35). Some values of v are given in Table 5. The focal ratio v decreases with increasing N because s in (32) decreases more rapidly then does β in (33). (But the focal length will increase with increasing N.)

TABLE 5

Beamwidth β, sky cover s, and focal ratio v, needed to reach N_{obs} sources/steradian and a total of 3000 sources.

N_{obs}	β	s	v
10^3	10!0	35°	9.9
10^4	3!2	3°5	5.6
10^5	1!0	21'	3.1
10^6	19"	2!1	1.8

8. Optimum solutions for reaching a given limit, N_{lim}

We call N_{lim} the number of sources per steradian which the antenna should be able to reach. For a given kind of receiver, we have, in principle, three free parameters to take care of this task: antenna surface, base area and wavelength. But this freedom is restricted by two economical conditions: we want to waste neither antenna surface nor base area. Once we have chosen the wavelength, the surface then is given by the brightness limit and the area is given by the resolution limit. This means that the task defines a one parameter family of optimum solutions.

Regarding the wavelength λ as the one free parameter, its range is restricted at both sides. One side is given by the single dish, because the base area cannot be made smaller than the antenna surface. Beginning at the single dish and moving to longer wavelengths, we gain smaller antenna surface and have to pay with an increase in base area. Finally, we reach the maximum of the brightness limit in figure 1, and this is the restriction at the other side, giving a minimum of antenna surface.

These optimum solutions lie at the crossing point of resolution limit and brightness limit where both noise contributions add up (see chapter 5). We therefore need

$$N_{res} = N_{vis} = 2\ N_{lim} \tag{37}$$

The antenna surface is $\pi\ a^2/4$, and from formula (13) and table 2 we get:

$$a = \left[\frac{2\ N_{lim}}{\psi\ (\lambda)} \right]^{1/3} \quad . \tag{38}$$

For the diameter of the base area we get from (24):

$$b = 13.0\ \lambda \sqrt{N_{lim}} \quad . \tag{39}$$

An antenna of the Mills cross type might be considered as one possible way to distribute the antenna surface over the base area. If we call d the width of the arms, the surface is 2bd and must equal $\pi\ a^2/4$. We thus have

$$d = 0.393\ a^2/b \quad . \tag{40}$$

III. THE EVALUATION OF N_{lim}

In Part II we have calculated which properties an antenna needs in order to reach any given limiting number of sources per steradian. It is our present concern to decide which value of N_{lim} should be reached. We want to be able to distinguish between different cosmological models with an antenna of minimum cost. Therefore, we need an estimate of the value of N_{lim} in the neighborhood of which the cosmological effects might become appreciable, and even a very rough estimate is better than none at all.

1. The sky brightness

Provided that $n \geq 1$, law (1) cannot continue infinitely toward fainter sources because the sky brightness then would be very great (Olbers paradox). Law (1) can be valid only down to a certain flux density S_{lim} in the neighborhood of which the slope n must decrease below 1. If we knew by observation the radio sky brightness produced by the background sources, we could estimate S_{lim} in a first approximation by integrating (1) to S_{lim} and neglecting the contribution beyond it. The above statements are true without regard to any cosmological model and without regard to the galactic or extragalactic nature of the background sources.

We call b_s the sky brightness due to the background sources, including sources from flux density S_{lim} on and up to an upper limit S_{max}:

$$b_s = \int_{S_{lim}}^{S_{max}} \frac{dN}{dS} S \, dS \ . \tag{41}$$

We use formula (5), putting $\lambda = 3.7$ m (to match some existing observations) and suppose that $S_{max} \gg S_{lim}$. Sky temperature and brightness are connected by $T_s = \lambda^2 b_s / 2k$ and we arrive at

$$S_{lim} = \left(0.0745 \times (34.2)^n \frac{n}{(n-1)\, T_s} \right)^{1/(n-1)} \times 10^{-26}\ Wm^{-2}\ cps^{-1}, \tag{42}$$

and finally,

$$N_{lim} = 15 \left(\frac{3.42 \times 10^{-25}}{S_{lim}} \right)^n = 15 \left(\frac{n-1}{n} \ \frac{T_s}{2.55} \right)^{\frac{n}{n-1}} \tag{43}$$

S_{lim} depends on the wavelength according to (2) and, in addition, both S_{lim} and N_{lim} might depend on the wavelength because of the effects of redshift. For out present purpose we neglect the latter and regard N_{lim} as the same for all wavelengths.

With regard to the observed sky brightness, however, the distinction between the galactic background radiation and the radiation of the isotropically distributed background of faint sources is extremely difficult to determine. Baldwin (1955) has measured the sky brightness at $\lambda = 3.7$ m for various galactic latitudes and longitudes. Assuming different galactic models, he could only give an upper limit of $\leq 500°$ K to the isotropic part of the background, and he compares this result with some estimates based on cosmological models yielding a lower limit of $\geq 200°$ K. In another approach, Mills and Slee measured the fluctuation (the bumpiness) of the background at high galactic latitudes at $\lambda = 3.5$ m and arrived at a rough estimate of $\approx 100°$ K for the background sources.

Table 6 is calculated from formula (43) within the limits of uncertainty of T_s as well as of n. The resulting uncertainty of N_{lim} is extremely high (see also figure 4). A fair guess might be to take $n = 1.5$ again and a medium temperature of, say, about 150°; this yields $N_{lim} = 10^5$ sources per steradian as the point where the deviations from a straight line should be large enough to be interesting.

2. Cosmological models

Another approach would be to calculate the log N, log S diagram for some different cosmological models and to see where they begin to differ. Very detailed calculations of this type have been performed by Priester (1958). In six figures he presents 48 different log N, Log S diagrams for comparsion, according to different assumptions (2 Hubble-constants x 3 space metrics x 4 mass parameters x 2 for collision or no collision

= 48). Summarizing, in the average, the deviations from a straight line become appreciable around 10^4 sources per steradian, and the differences between the different models begin to show up around 10^5 and get quite large around 10^6. A lower value for N_{obs} therefore should lie between 10^5 and 10^6 sources per steradian.

TABLE 6

The upper limit for a straight-line log N, log S diagram, for different values of slope n and sky temperature T_s

n	N_{lim} (sources/sterad.)			
	T_s = 50°	100°	200°	500°
1.3	1.0×10^4	2.1×10^5	4.2×10^6	2.2×10^8
1.5	4.2×10^3	3.4×10^4	2.7×10^5	4.2×10^6
1.7	2.4×10^3	1.3×10^4	7.0×10^4	6.5×10^5
2.0	1.4×10^3	5.7×10^3	2.3×10^4	1.4×10^5

These calculations were performed under the assumption of one uniform standard source of average absolute magnitude. In reality, however, one ought to integrate over the luminosity function of radio sources, which spreads over a very large range of about 14 magnitudes. This integration results in a "smearing out" of the cosmological effects over a larger range: the deviations from a straight line will begin at much lower numbers, and real large differences between cosmological models will occur at much higher numbers. But the lower value for N_{lim} should not be changed too much by this integration.

Minkowski (1961) has constructed a luminosity function from the identified sources and has calculated a log N, log S diagram under the following assumptions: Hubble constant = 75 (km/sec)/Mpc, deceleration parameter $q = -R \ddot{R}/\dot{R}^2 = 1$, cosmological term $\Lambda = 0$. His result is that the slope, n, should be considerably less than 1.5 already in the range of the present surveys, for example n = 1.26 at N = 20. Using Minkowski's formulae we have extended the calculation to lower flux densities, as shown in figure 4. The values of n are written along the curve. For comparsion, one point for the Einstein-de Sitter model is calculated.

Included in figure 4 are some straight lines of different slopes, illustrating the present range of observational uncertainty. The points of table 6 are marked on each line and connected by broken lines, indicating about where the slope should decrease below 1 according to different values of n and of the sky temperature. The integration of Minowski's model yields a sky temperature of $T_s = 109°$ at $\lambda = 3.7$ m, and the point where n = 1 lies at N = 7.0×10^5. Entering formula (43) with T = 109° and n = 1.26, we arrive at $N_{lim} = 5.7 \times 10^5$, in good agreement.

3. Results

Both estimates, sky brightness and models, give the same result: N_{lim} should lie between 10^5 and 10^6 sources per steradian. We therefore adopt this conclusion: for a successful attack on the cosmological problem the antenna should be able to reach

$$N_{lim} = 3 \times 10^5 \text{ sources per steradian} . \quad (44)$$

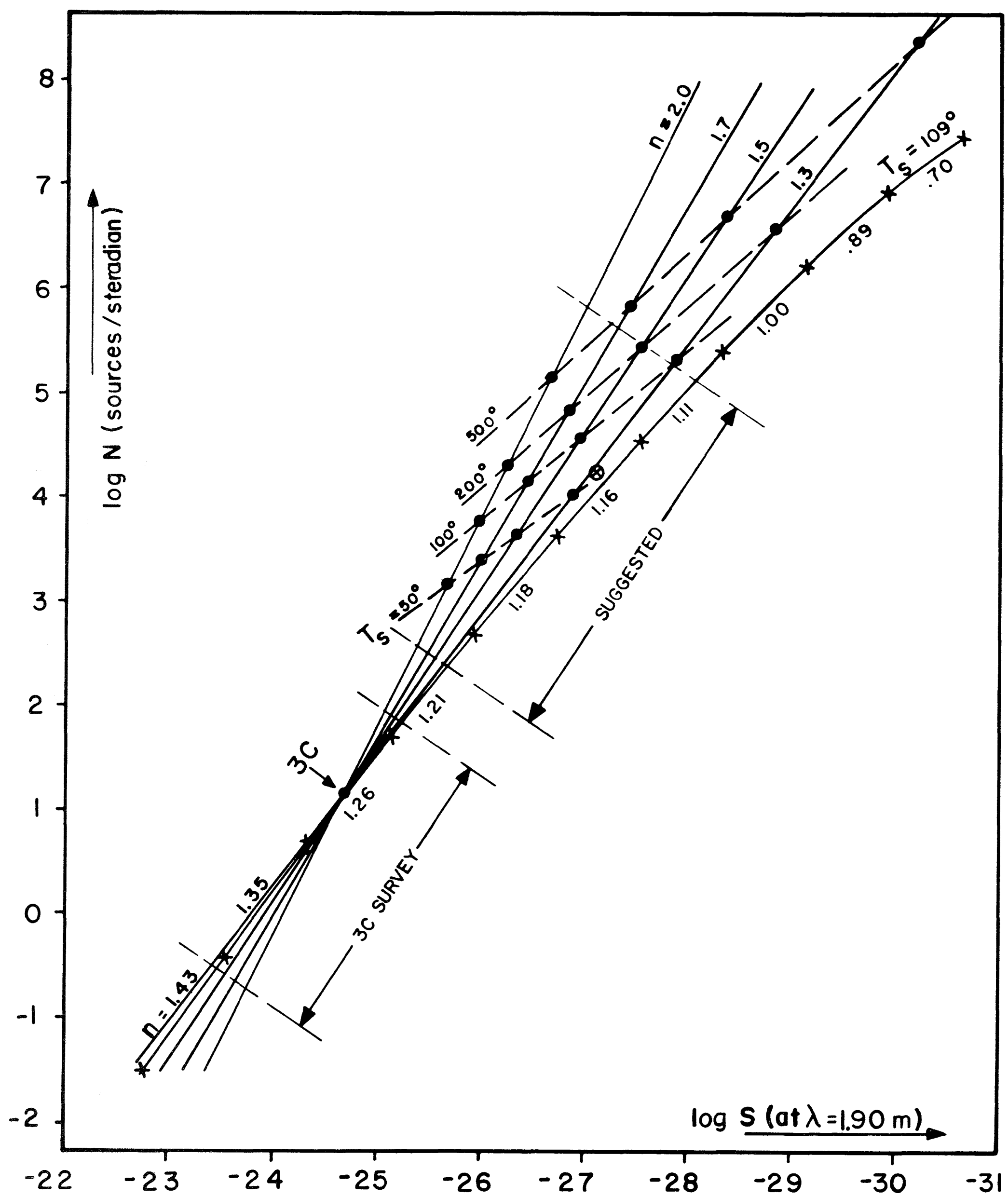

Fig. 4. The evaluation of N_{lim}. The straight lines of slope 1.3 ... 2.0 illustrate the present range of observational uncertainty. The dashed lines connect the points of table 6, in the neighborhood of which the slopes must decrease below 1 for different sky temperatures. The slightly curved line with the crosses represents a theoretical expectation by Minkowski, the values of n are written along the curve. ⊗=One point of Einstein de Sitter model.

In (31) we adopted a total of M = 3000 sources. If we admit that statistics usually begin at 3, the proposed survey then covers a range of three powers of ten, reaching down to a lower limit of

$$N = 300 \text{ sources per steradian} \tag{45}$$

After having adopted a fixed value for N_{lim}, we apply formulae (38), (39) and (40). The resulting one-parameter family of optimum solutions is shown in table 7, for both present receivers and a maser of 20° noise temperature.

TABLE 7

Optimum solutions for reaching $N_{lim} = 3 \times 10^5$ sources/steradian
$\pi a^2/4$ = collecting surface of antenna system
b = diameter of base area
d = width of arms in case of Mills cross

	present receivers				maser, 20°			
	λ	a	b	d	λ	a	b	d
	cm	m	m	m	cm	m	m	m
round dish	8	565	565	-	2.1	146	146	-
intermediate	12	467	855	100	3.5	115	249	21.1
	18	397	1280	48	5.2	99	370	10.5
	27	342	1920	23.9	7.7	87.1	548	5.4
	40	298	2850	12.2	11	80.0	783	3.21
	60	264	4270	6.4	17	75.4	1210	1.84
	90	239	6410	3.5	24	72.8	1710	1.22
minimum surface	135	227	9610	2.1	36	71.5	2560	0.79

Finally, we insert N_{lim} from (44) into equations (32), (33) and (36) and obtain:

$$\begin{aligned} &\text{sky coverage } s = 6.9 \text{ min of arc} \\ &\text{beamwidth } \beta = 34.7 \text{ sec. of arc} \\ &\text{focal ratio } v = 2.37 \end{aligned} \tag{46}$$

It might be necessary, however, to introduce a safety factor at this point. The values (46) would give a total of 3000 sources if the slope n = 1.5 would continue until N_{lim}. On the other side N_{lim} was defined as the point where n should deviate appreciably from 1.5, and if n is less we would get less than 3000 sources, for example only 300 in case of Minkowski's model of figure 4. Thus we should increase the sky coverage by a safety factor of about ten, although this might increase the price of the antenna considerably.

The situation somewhat improves if we consider an instrument which can observe at various wavelengths. If it turns out that the slope is much lower than assumed, we can choose a longer wavelength because we then would need less resolution and more sensitivity. A rough estimate shows that it still might be prudent to introduce a safety factor of about three. Instead of (46) we therefore suggest

$$\begin{aligned} &\text{sky coverage} \quad s = 20 \text{ min. of arc} \\ &\text{focal ratio} \quad v = 4 \end{aligned} \tag{47}$$

IV. EXISTING INSTRUMENTS

In Part III an estimate was given for N_{lim}, the limiting number of sources per steradian which an antenna should reach to make a reliable distinction between different cosmological models. Our present question is whether one of the already existing instruments, or one of the instruments under construction, could reach this limit.

Table 8 shows a number of larger instruments of both categories. We want to emphasize that the figures in the table are supposed to give only an estimate of the order of magnitude; no great accuracy is claimed, and these should be regarded as upper limits. We have always taken b as equal to the full size of the antenna, whereas the tapering of the illumination, needed for decreasing the sidelobes, would give a lightly shorter value for b and therefore a smaller number of sources. The users of aperture synthesis theorize that their resolution limit is the same as that of a dish of the size of their base area, but some doubts might be raised as to whether this still is true for very high number per steradian where a large number of different antenna positions must be used and a very elaborate process of data analysis is needed. Nevertheless, we have calculated under the assumption that this theoretical accuracy is correct.

All calculations have been made under the following assumptions: $n = 1.5$, $S \approx \lambda^{0.8}$, bandwidth = 5% of frequency, integration time = 10 sec, signal/noise = 5 at brightness limit, 75 beam solid angles per source at resolution limit (which means signal/noise = 5 regarding background sources.)

The table gives the location and the type of the antenna, followed by the limiting wavelength, λ_0, down to which the surface can be used. N_{obs} is the maximum number of sources per steradian, and λ gives the wavelength where this number has its maximum. A hyphen in this column indicates that the antenna is resolution limited by too large a value of λ_0, and that N_{obs} therefore is calculated at λ_0. In this case, any improvement of the receiver, for example by taking a maser, would not increase N_{obs} because of the resolution limit.

A value of 20° is assumed for the noise temperature of a good maser of the future. If the surface is accurate enough (λ_0 small), the maximum of N_{obs} can be reached, and this maximum and the wavelength belonging to it are given in the next two columns. A hyphen again indicates that this optimum wavelength would be smaller than λ_0 and that N_{obs} therefore is calculated at λ_0.

The last column gives a rough figure for the price of the antenna where available. If a maser were applied, the price of the maser should be added. The price of a maser of the assumed quality might be of the order of $200,000.

The table shows the extreme importance of high surface accuracy, because no further improvement is possible once the resolution limit is reached, and because the maximum efficiency always lies at relatively short wavelengths. The larger the antenna, the shorter is this optimal wavelength (see table 4). The restriction toward shorter wavelengths, given by the slow atmospheric fluctuations, is already included in the calculations according to (10), assuming short integration times of 10 sec.

TABLE 8

Upper limits for the maximum number of observable sources per steradian for various instruments

1. In Operation

Location	Type	surface limit λ_0 cm	present rec. N_{obs} sterad^{-1}	present rec. λ cm	best maser N_{obs} sterad^{-1}	best maser λ' cm	approx. price 10^6 $
Caltech, Calif.	2 parab. 90 ft., max. sep. 500 m	10	1 300	90	40 000	18	1.3
Carnegie Inst., Wash. DC	paraboloid 60 ft.	3	38	23	1 200	3.8	0.2
Green Bank, W. VA.	paraboloid 85 ft.	3	93	22	3 200	3.5	0.3
CSIRO, Australia	Mills cross, 1500 ft.	350	200	-	-	-	
Manchester, Engld.	paraboloid 250 ft.	30	760	-	-	-	1.0
Manchester, Engld.	Earth par. 218 ft.	21	1 200	-	-	-	
Mullard Obs. Cambridge	movable interf. 2300 ft. separ.	170	2 000	-	-	-	0.1
Leningrad, USSR	par. strip 120 x 3 m	3.2	56	23	2 000	3.9	
Leningrad, USSR	paraboloid 22 m	0.8	60	23	2 200	3.8	

2. Under construction

Location	Type	λ_0 cm	N_{obs} sterad^{-1}	λ cm	N_{obs} sterad^{-1}	λ' cm	10^6 $
CSIRO, Australia	parabol. 210 ft.	10	1 000	10	4 800	-	1.8
Lebedev I, USSR	Mills cross 1000 c 40 m	150	5 200	-	-	-	20
Danville, Ill.	Earth par. cyl. 600 x 400 ft.	75	600	-	-	-	0.5
Delaware, Ohio	par. seg. 360x70 ft.	21	600	-	-	-	0.5
Green Bank W. Va.	parabol. 140 ft.	1	340	18	12 000	3.1	8
Cornell Univ. Puerto Rico	Earth spheriod 1000 ft.	30	8 000	-	-	-	4.5
Sugar Grove W. Va.	Paraboloid 600 ft.	10	15 000	11	30 000	-	150

Table 8 gives the answer to our present question. Even if we assume the upper limits of N_{obs} represent the actual number of observable sources per steradian, and even in case of the largest instruments, with best masers a factor of about ten is still lacking. According to our estimates in Part III it should be possible to detect deviations from a straight line in the log N, log S diagram, but the range where different cosmological models show appreciable differences is not likely to be reached by the instruments which are in operation or under construction.

Conclusions

As one of the most challenging tasks for a very large antenna we have defined: to reach so many sources per steradian that reliable decisions between different cosmological models become possible. In Part II we have calculated, in general and with special precaution, which antenna properties are needed to reach a given number of sources per steradian, and in Part III we have estimated N_{lim}, the limiting number which should be reached. This number is three and a half powers of ten higher than the upper limit of the 3C survey, and even the largest radio telescopes now under construction will miss it by at least a factor of ten.

In spite of this fact, however, the solutions of table 7 show that this task is not out of practical reach, not even for conventional methods, if one keeps to one of the optimum solutions with their relatively short wavelengths. A fixed round dish of 150 m diameter usable down to a wavelength of 2 cm, with movable feed and maser, is large indeed but not impossible. The same is true for a Mills cross of 2 km length and 24 m width, working at 27 cm with present vacuum tube receivers. Instruments of this size would be more expensive than present ones, but not overwhelmingly so.

These two examples show the possibility of reaching our goal by adjusting one or the other present type of antenna to the special purpose we have in mind, but we might reach the goal much more cheaply if a special type of antenna could be developed for this very purpose. A suggestion of this kind will be made in a following paper.

It is a pleasure to express my thanks to the whole staff of the NRAO for many beneficial discussions.

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